Not All Writing Desks are Ravens: An Introduction to Topology of Surfaces

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Outline

1. Stating the Problem
2. Equivalent Surfaces
   2.1 Some Topology
   2.2 Examples of Surfaces
   2.3 Classifying Compact Surfaces
3. A Solution?

A Bit of History

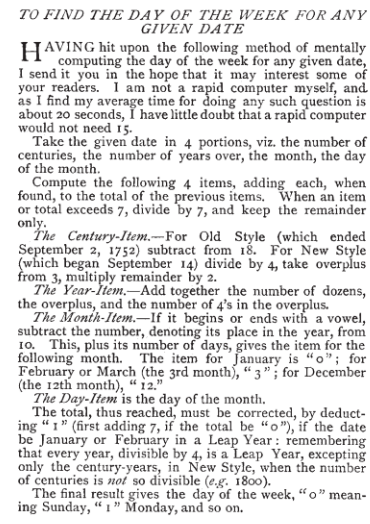
- Lewis Carroll (Charles L. Dodgeson)
  Born in Cheshire in 1832.
  Died in Guildford in 1898.
- Double first in Mathematics from Christ Church, Oxford.
- Deacon, lecturer, librarian, and published researcher.

Alice Through the Looking Glass
Jeanne Argent

His Papers

Three papers in *Nature*

- ‘To find the Day of the Week for any Given Date’
- ‘Brief Method of Dividing a Given Number by 9 or 11’ (As Dodgeson)
- ‘Abridged Long Division’



TO FIND THE DAY OF THE WEEK FOR ANY GIVEN DATE

HAVING hit upon the following method of mentally computing the day of the week for any given date, I send it you in the hope that it may interest some of your readers. I am not a rapid computer myself, and as I find my average time for doing any such question is about 20 seconds, I have little doubt that a rapid computer would not need 15.

Take the given date in 4 portions, viz. the number of centuries, the number of years over, the month, the day of the month.

Compute the following 4 items, adding each, when found, to the total of the previous items. When an item or total exceeds 7, divide by 7, and keep the remainder only.

The Century-Item.—For Old Style (which ended September 2, 1752) subtract from 18. For New Style (which began September 14) divide by 4, take overplus from 3, multiply remainder by 2.

The Year-Item.—Add together the number of dozens, the overplus, and the number of 4's in the overplus.

The Month-Item.—If it begins or ends with a vowel, subtract the number, denoting its place in the year, from 10. This, plus its number of days, gives the item for the following month. The item for January is "0"; for February or March (the 3rd month), "3"; for December (the 12th month), "12."

The Day-Item is the day of the month.

The total, thus reached, must be corrected, by deducting "1" (first adding 7, if the total be "0"), if the date be January or February in a Leap Year: remembering that every year, divisible by 4, is a Leap Year, excepting only the century-years, in New Style, when the number of centuries is *not* so divisible (*e.g.* 1800).

The final result gives the day of the week, "0" meaning Sunday, "1" Monday, and so on.

The Question

Why is a Raven like a Writing Desk? - Mad Hatter (Alice in Wonderland 1865)

Previous Answers

- Because (Edgar Allen) Poe wrote on both. [Sam Loyd]
- The higher the fewer. [Stephen King - The Shining]

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- Neither are ice cream.

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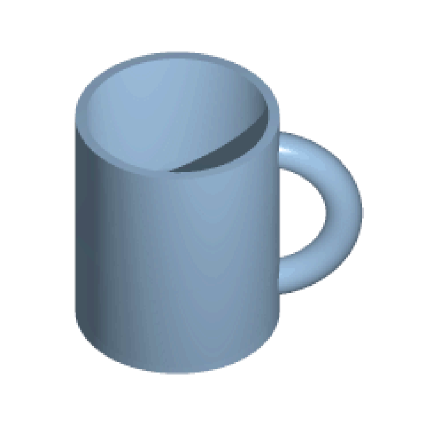
These are not mathematically precise - we can do better!

Doughnuts and Coffee Cups

Topologists have a similar problem - they cannot distinguish between doughnuts and coffee cups!

A coffee cup can be continuously deformed into a doughnut and vice versa, when they are interpreted as surfaces. Let's see why...



Topological Spaces

A *topological space* is a set X equipped with a collection τ of (open) subsets of X, such that

- $\emptyset \in \tau$ and $X \in \tau$
- $U_i \in \tau$ for all $i \in \{1, 2, \cdots n\}$, $n \in \mathbb{N}$ then $\bigcap_{i=1}^{n} U_i \in \tau$
  (Finite intersections of open sets are open)
- $U_i \in \tau$ for all $i \in I$ then $\bigcup_{i \in I} U_i \in \tau$
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A topological space is called *Hausdorff* if, for any two $x, y \in X$ with $x \neq y$, there exists open subsets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Euclidean Topology

The *Euclidean topology* on $\mathbb{R}^m$ has open sets given by unions and intersections of open balls

$$B(x,r) = \{y \in \mathbb{R}^m \,|\, d(x,y) < r\}$$

where $x \in \mathbb{R}^m$ and $r > 0$.

The map $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is the Euclidean metric

$$d(x,y) = \sqrt{\sum_{i=1}^{m} (y^i - x^i)^2}$$

The Euclidean topology is Hausdorff.

Maps Between Topological Spaces

Let (X, τ) and (Y, σ) be topological spaces.

A *homeomorphism* $f : X \to Y$ is a bijective function which preserves topology:

- All open sets in X are mapped by f to open sets in Y.
- All open sets in Y are mapped by f^{-1} to open sets in X.

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Equivalently, a homeomorphism is a bijective function where both f and f^{-1} are continuous in the appropriate sense.

Homeomorphic Spaces

Two topological spaces are called *homeomorphic* if there exists a homeomorphism between them.

Being homeomorphic is an equivalence relation:

- (Reflexivity) A space is homeomorphic to itself, with f the identity.
- (Symmetry) If $f : X \to Y$ is a homeomorphism, so is $f^{-1} : Y \to X$.
- (Transitivity) If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms then $g \circ f : X \to Z$ is a homeomorphism.

Surfaces

A (topological) *surface* is a Hausdorff topological space X, such that for each point $x \in X$, there exists an open set $U \subseteq X$, which is homeomorphic to an open subset of $\mathbb{R}^2$ with Euclidean topology.

Surfaces *'look locally like* $\mathbb{R}^2$*'* so are two dimensional objects.

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Surfaces *'look locally like* $\mathbb{R}^2$*'* so are two dimensional objects.

A surface is called *compact* if, for every collection of open sets which covers the surface entirely, there is a finite sub-collection which also covers the whole surface.

Embeddings and Subspace Topology

- Let (Y, σ) be a topological space with A a subset of Y. The *subspace/ induced topology* on A is defined by

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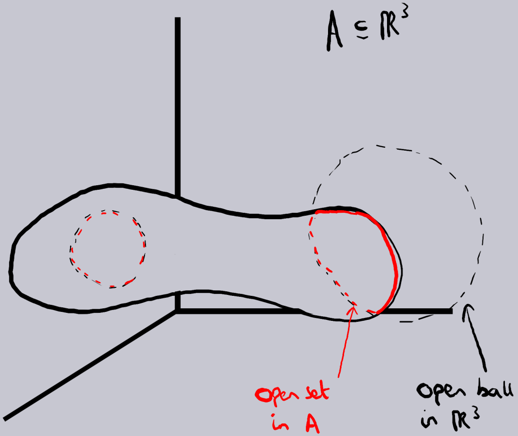
- For (X, τ) a topological space and (A, σ_A) a subspace of (Y, σ): If there exists a homeomorphism $f : (X, \tau) \to (A, \sigma_A)$, then (X, τ) is said to be (continuously) *embedded* in (Y, σ).
- In this case, one can consider (A, σ_A) in place of (X, τ).

Subspace Topology in $\mathbb{R}^m$

Here $\mathbb{R}^3$ has the Euclidean topology consisting of open balls.



A Note on Embeddings

- Recall a surface is defined abstractly - it does not necessarily live in another space.
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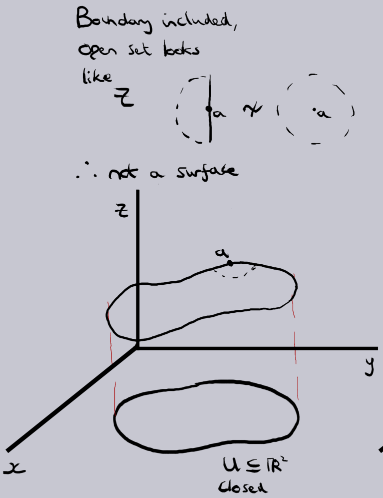
- While some surfaces can be embedded in $\mathbb{R}^3$, this is not true in general.
- (Whitney Embedding Theorem) Smooth (manifold) surfaces can be smoothly embedded in $\mathbb{R}^4$.
- For surfaces embedded in $\mathbb{R}^m$, compactness is equivalent to closed boundedness (Heine–Borel Theorem).

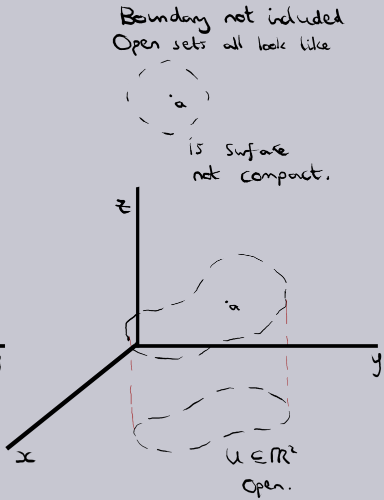
Non-examples of Compact Surfaces

- $\mathbb{R}^2$ itself is a surface, but is not compact as it is not bounded (it is closed, however).
- The Möbius strip (without boundary) is a surface, but is not compact.
- A graph over a closed subset of $\mathbb{R}^2$ is not a surface as for points on the boundary, no open neighbourhood looks like an open subset of $\mathbb{R}^2$.
- A graph over an open subset of $\mathbb{R}^2$ is a surface, but is not closed and therefore not compact.

Non-examples of Compact Surfaces

Let $U \subseteq \mathbb{R}^2$ and consider the graph of a function $f : U \to \mathbb{R}$.





The Basic Compact Surfaces

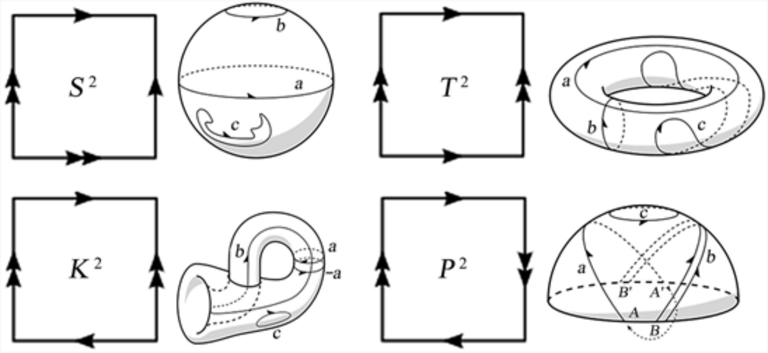
- 2-Sphere $\mathbb{S}^2$.
- 2-Torus $\mathbb{T}^2$.
- Real Projective Plane $\mathbb{RP}^2$.

$\mathbb{S}^2$ and $\mathbb{T}^2$ both embed in $\mathbb{R}^3$ and are closed and bounded, hence compact.

$\mathbb{RP}^2$ embeds smoothly in $\mathbb{R}^4$ (not $\mathbb{R}^3$) and is closed and bounded there.

Compact Surfaces

Basic compact surfaces are obtained by identifying sides of a square:



Connected Sums

Compact surfaces can be constructed from the previous basic objects via the following *surgery*:

- Take two compact surfaces and remove an open disc from each.
- Join the two openings via an open ended cylinder.

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The connected sum of two copies of $\mathbb{RP}^2$ is a Klein Bottle.

The connected sum of two $\mathbb{T}^2$ is a compact surface with two holes.

All of the Compact Surfaces

All compact surfaces are homeomorphic to a sphere, a connected sum of toruses, or a connected sum of projective planes.

Some comments:

- Removing an open disc from $\mathbb{S}^2$ leaves a closed disc, so the connected sum of a compact surface and $\mathbb{S}^2$ leaves the former unchanged.
- The connected sum of $\mathbb{T}^2$ and $\mathbb{RP}^2$ is homeomorphic to the connected sum of three copies of $\mathbb{RP}^2$.

Orientable Surfaces

A surface X is orientable if no open subset of X is homeomorphic to the Möbius strip.

Intuitively, this says that if you pick a point on a surface and travel in any fixed direction, if you return to your initial point, you will be facing the same way.

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Both $\mathbb{S}^2$ and $\mathbb{T}^2$ are orientable, while $\mathbb{RP}^2$ is a Möbius strip with a disc glued to its edge and is therefore not orientable.

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- Hence, a connected sum of compact surfaces embeds into $\mathbb{R}^3$ if and only if the individual surfaces embed in $\mathbb{R}^3$.
- Surfaces homeomorphic to the sphere or connected sum of toruses embed into $\mathbb{R}^3$, while connected sums of $\mathbb{RP}^2$ do not.

Distinguishing Compact Surfaces

We know whether or not a compact surface is homeomorphic to a connected sum of $\mathbb{RP}^2$ depending on if it can be realised in $\mathbb{R}^3$ or not.

We now need a way of distinguishing between compact surfaces which are homeomorphic to the sphere and those which are homeomorphic to connected sums of toruses.

Genus of a Compact Surface

The genus of a compact surface embedded in $\mathbb{R}^3$ is loosely defined by the number of holes the surface has.

More precisely, the genus corresponds to the maximum number of simple, closed, disjoint curves (circles) that can be drawn on the surface without disconnecting it.

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List of genera:

- Genus of $\mathbb{S}^2$ is 0
- Genus of $\mathbb{T}^2$ is 1
- Genus of connected sum of n copies of $\mathbb{T}^2$ is n

Classification of Compact Surfaces

A compact surface embedded in $\mathbb{R}^3$*, is defined up to homeomorphism by its genus.*

That is, the genus is preserved under homeomorphism, hence two compact surfaces are homeomorphic if and only if they have the same genus.

We can therefore determine whether a homeomorphism exists between compact surfaces by calculating their genera.

Doughnuts and Coffee Cups 2

In particular, a coffee cup has genus 1, hence must be homeomorphic to $\mathbb{T}^2$, which describes a ring doughnut. (A coffee cup is not homeomorphic to a Berliner, which has genus zero)

Modelling a Raven

So what is the genus of a raven?

- First we must assume that the raven is hollow, such that it can be modelled by a compact surface (it is finite and closed).
- The nervous/ cardiovascular systems would be in the hollowed regions, so we can disregard these.

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So a raven has genus 1!

Genus of a Writing Desk

Genus 0

Genus 1

Genus 3

This Writing Desk is a Raven!

Extension of Genus

... Genus can be extended to open surfaces by patching their boundaries with a disc.

... It can be extended to non-orientable surfaces...

... One can define the Euler characteristic also...