Geometric Techniques in PDE Theory and Fluid Dynamics

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Preamble

1. Monge–Ampère Geometry

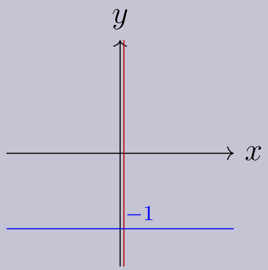
2. Geometry of 2D Incompressible Fluid Flows

3. Towards Higher Monge–Ampère Equations

Conclusions

PDEs as Manifolds

- *k-th Jet Bundle* $J^k(M,N)$ is space of all possible values of $x, y, D^1 y, \cdots D^k y$ [Ehresmann 1951, Bryant et al. 1991]
- k-th order PDE $F(x, y, D^1 y, \cdots D^k y) = 0$ can be seen as the space $\mathcal{E} \subset J^k(M,N)$ of points satisfying equation.
- Solutions $\psi : M \to N$ are submanifolds $L \subset \mathcal{E}$, e.g. $F(x, \psi(x), D^1\psi, \cdots D^k\psi) = 0$.
- Properties of geometry tell us about properties of equation and solutions.



$3x^2(1+y) = 0$ consists of two lines, each of which is a solution.

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Outline of Talk

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1. Monge–Ampère Geometry

What are Monge–Ampère Equations?

- ***MAE***: non-linear, second-order PDE, given by quasi-linear combinations of the minor determinants of the ***Hessian*** of ψ:

$$\mathrm{Hess}(\psi) = \begin{pmatrix} \psi_{x^1x^1} & \psi_{x^1x^2} & \cdots & \psi_{x^1x^n} \\ \psi_{x^2x^1} & \psi_{x^2x^1} & \cdots & \psi_{x^2x^n} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{x^nx^1} & \psi_{x^nx^2} & \cdots & \psi_{x^nx^n} \end{pmatrix}$$

- ***Quasi-Linear***: coefficients can depend on x, ψ and $D^1\psi$ non-linearly.
- ***k-th Minor Determinant***: determinant of the $k \times k$ sub-matrix with entries given by intersections of k rows and columns.

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MAEs in Two Dimensions

- In two dimensions, MAEs take the form

$$A\psi_{x^1x^1} + 2B\psi_{x^1x^2} + C\psi_{x^2x^2} + D\left(\psi_{x^1x^1}\psi_{x^2x^2} - \psi^2_{x^1x^2}\right) + E = 0\,.$$

  where $A, B, \dots E$ can depend on $x^1, x^2, \psi, \psi_{x^1}, \psi_{x^2}$ non-linearly.

- If $A, B, \dots E$ do not depend on ψ, we have a *Symplectic* MAE.
- Symplectic MAEs can be encoded in T^*M rather than $J^2(M, N)$. where M is the *Configuration Space*.

Some Examples You May Know

- 2D Reaction-Diffusion: $\psi^\alpha \psi_{xx} + [\alpha \psi^{\alpha-1}\psi_x - \psi_t + F(\psi)] = 0.$
- 3D Chynoweth–Sewell: $[\psi_{xx}\psi_{yy} - (\psi_{xy})^2] + \psi_{zz} = 0.$
- 4D Khokhlov–Zabolotskaya: $\psi_{tt} + \psi_{yy} + \psi_{zz} - \psi_{xt} + (\psi_t)^2 = 0.$

- Laplace: $\Delta\psi := \psi_{x^1x^1} + \psi_{x^2x^2} + \cdots + \psi_{x^nx^n} = 0.$
- Wave: $\Box\psi := \psi_{tt} - \psi_{x^1x^1} - \psi_{x^2x^2} - \cdots - \psi_{x^nx^n} = 0\ .$

Geometry to Equation: A Quick Example

Consider a 2-form on $T^*\mathbb{R}^2$ (with coordinates x^1, x^2, q_1, q_2):

$$\alpha = \mathsf{d}q_1 \wedge \mathsf{d}x^2 - \mathsf{d}q_2 \wedge \mathsf{d}x^1 \,.$$

Define $L_\psi \coloneqq \{(x^1, x^2, \psi_{x^1}, \psi_{x^2})\} \subset T^*\mathbb{R}^2$ (fix q_1 and q_2 at each x).

$$\begin{aligned} \alpha|_{L_\psi} &= \mathsf{d}(\psi_{x^1}) \wedge \mathsf{d}x^2 - \mathsf{d}(\psi_{x^2}) \wedge \mathsf{d}x^1 \\ &= (\psi_{x^1x^1} + \psi_{x^2x^2})\, \mathsf{d}x^1 \wedge \mathsf{d}x^2 \end{aligned}$$

So $\alpha|_{L_\psi} = 0$ if and only if $\Delta\psi = 0$, i.e. ψ solves $\Delta\psi = 0$.

Symplectic Forms and Non-Uniqueness

A *Symplectic* form ω on $T^*\mathbb{R}^m$ is

- a 2-form: skew-symmetric and bilinear,
- *Closed*: $\mathrm{d}\omega \equiv 0$,
- *Non-Degenerate*: $\omega(X, \cdot) \equiv 0$ if and only if $X \equiv 0$.

The canonical choice is

$$\omega = \mathrm{d}q_i \wedge \mathrm{d}x^i = \begin{pmatrix} 0_m & -I_m \\ I_m & 0_m \end{pmatrix}$$

Then $\omega|_{L_\psi} = 0$ is trivial, so $\alpha|_{L_\psi} = 0$ and $(\alpha + F(x,q)\,\omega)|_{L_\psi} = 0$ are the same equation! Which one do we pick?

Effective Forms and Equivalence Classes

- An m-form α on $T^*\mathbb{R}^m$ is called *ω-Effective* if $\alpha \wedge \omega = 0$.
- For symplectic form ω, every m-form β on $T^*\mathbb{R}^m$ decomposes as

$$\beta = \alpha + \omega \wedge \beta_0 \, ,$$

  for some unique $(m-2)$-form β_0 and ω-effective m-form α [Hodge–Lepage–Lychagin].
- This defines equivalence classes $[\alpha]$ where the only effective form is α and $\beta|_{L_\psi} = 0$ is equivalent to $\alpha|_{L_\psi} = 0$.

Monge–Ampère Structures

- A *Monge–Ampère Structure* on $T^*\mathbb{R}^m$ is a pair (ω, α), where ω is a symplectic form and α is an ω-effective m-form [Banos 2002].
- In 2D with ω canonical, the ω-effective forms are

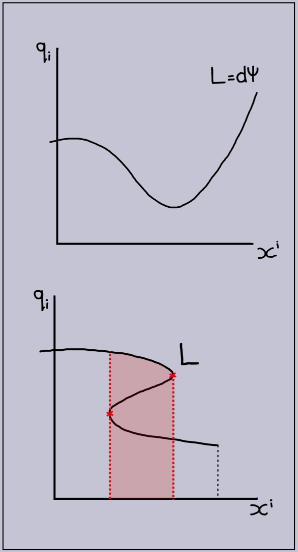
$$\alpha = A\,\mathrm{d}q_1 \wedge \mathrm{d}x^2 + B\left(\mathrm{d}x^1 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}x^2\right) + C\,\mathrm{d}x^1 \wedge \mathrm{d}q_2 + D\,\mathrm{d}q_1 \wedge \mathrm{d}q_2 + E\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

- These α are in bijection with MAEs: $\alpha|_{L_\psi} = 0$ is precisely

$$A\psi_{x^1x^1} + 2B\psi_{x^1x^2} + C\psi_{x^2x^2} + D\left(\psi_{x^1x^1}\psi_{x^2x^2} - \psi_{x^1x^2}^2\right) + E = 0\,.$$

Classical and Generalised Solutions

- A *Classical Solution* $\psi \in \mathcal{C}^\infty(\mathbb{R}^m)$ corresponds to $L_\psi = \{x, D^1\psi(x)\}$ with $\alpha|_{L_\psi} = 0$.
- A *Generalised Solution* of a MAS is an m-dimensional submanifold $L \subset T^*\mathbb{R}^m$ s.t. $\omega|_L = 0$ and $\alpha|_L = 0$.
- If projection $\pi : L \to \mathbb{R}^m$ is not
  – surjective, ψ not defined on whole domain.
  – injective, ψ is multivalued [Vinogradov 1970].
  – immersive, ψ is singular [Arnold 1990].



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The Pfaffian (2D)

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- The *Pfaffian* is defined by $\alpha \wedge \alpha =: f_\alpha \omega \wedge \omega$ where $f_\alpha = AC - B^2 - DE$.
- Here, f_α is the determinant of the coefficient matrix of the linearisation of $\alpha|_{L_\psi} = 0$.
- Hence, the Monge–Ampère equation $\alpha|_{L_\psi} = 0$ is

  elliptic $\Leftrightarrow f_\alpha > 0$.

  hyperbolic $\Leftrightarrow f_\alpha < 0$.

  parabolic $\Leftrightarrow f_\alpha = 0$.

Almost (Para-)Complex Structure (2D)

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- Set $\tilde{\alpha} = \frac{1}{\sqrt{|f_\alpha|}}\alpha$, so that $f_{\tilde{\alpha}} = \mathrm{sign}(f_\alpha)$.
  Up to sign, multiples of α have the same $\tilde{\alpha}$ (removes scaling).
- Define endomorphism of vector fields $J : \mathfrak{X}(T^*\mathbb{R}^2) \to \mathfrak{X}(T^*\mathbb{R}^2)$ by

$$\tilde{\alpha}(\cdot\,,\cdot) =: \omega(J\,\cdot\,,\cdot) \quad \left(J = \omega^{-1}\tilde{\alpha} \text{ as matrices}\right)\ ,$$

- $f_\alpha \lessgtr 0 \Leftrightarrow J^2 = \pm I_4$ and $\mathrm{tr}(J) = 0$ [Lychagin et al. 1993]

The Lychagin–Rubtsov Theorem (2D)

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MAEs are *locally equivalent* if there exists a (local) symplectomorphism $F : (T^*\mathbb{R}^2, \omega, \alpha_1) \to (T^*\mathbb{R}^2, \omega, \alpha_2)$, i.e.

$$F^*\omega = \omega \text{ and } F^*\alpha_2 = \alpha_1 \, .$$

The following conditions are equivalent [Lychagin et al. 1993]:

- $\alpha|_{L_\psi} = 0$ is locally equivalent to $\Box\psi = 0$ or $\Delta\psi = 0$.
- $\mathrm{d}(\tilde{\alpha}) = 0$ (with $f_\alpha \lessgtr 0$).
- J is integrable (with $J^2 = \pm I_2$).

These criteria do not always hold.

Lychagin–Rubtsov Metrics (2D)

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- Picking a non-degenerate, ω-effective, and α-effective 2-form K, we can define a symmetric, bilinear form

$$\hat{g}(\cdot\,,\cdot) := -K(J\cdot\,,\cdot)$$

called a Lychagin–Rubtsov metric [Napper et al 2023].

- Up to conformal scaling of K, a choice of $\hat{g}$ corresponds to a choice of almost (pseudo-)quaternionic structure on $T^*\mathbb{R}^2$. There is an $Sp(1) \cong S^3$ of choices [Thesis].

2. Geometry of 2D Incompressible Fluid Flows

Incompressible Navier–Stokes

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- Homogeneous, Incompressible Navier–Stokes on $\mathbb{R}^m$

$$\begin{aligned} \partial_t v^j &= -v^i \nabla_i v^j - \nabla_j p + \nu \Delta v^j \; (-c_j)\,, \\ \nabla_i v^i &= 0\,. \end{aligned}$$

- Taking the divergence of the first and applying the second:

$$\zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} = \Delta p \; (+\nabla_i c^i)\,.$$

where $\zeta_{ij} = \frac{1}{2}(\nabla_j v_i - \nabla_i v_j)$ is the vorticity form
and $S_{ij} = \frac{1}{2}(\nabla_j v_i + \nabla_i v_j)$ is the strain-rate tensor.

Pressure Equation in Two Dimensions

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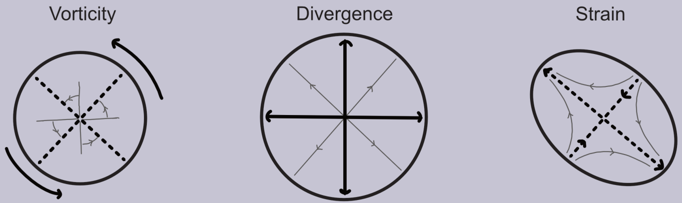
- In 2D, there exists a *stream function* $\psi \in \mathscr{C}^\infty(\mathbb{R}^2)$ such that $v^1 = -\psi_{x^2}$ and $v^2 = \psi_{x^1}$.
- Function dictates the direction of a particle dropped into the flow.
- Substituting this into Navier–Stokes, $\nabla_i v^i = 0$ is trivially satisfied and the pressure equation becomes an MAE for ψ:

$$\Delta p = 2\left(\psi_{x^1x^1}\psi_{x^2x^2} - (\psi_{x^1x^2})^2\right) .$$

Known Result: The Q-Criterion

- $\zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} = \Delta p = 2\left(\psi_{x^1x^1}\psi_{x^2x^2} - (\psi_{x^1x^2})^2\right).$
- Q-criterion [Weiss 1991, Larchevêque 1993]:
  Vorticity dominates $\Leftrightarrow$ $\Delta p > 0$ $\Leftrightarrow$ *Elliptic equation.*
  Strain dominates $\Leftrightarrow$ $\Delta p < 0$ $\Leftrightarrow$ *Hyperbolic equation.*
  No dominance $\Leftrightarrow$ $\Delta p = 0$ $\Leftrightarrow$ *Parabolic equation.*



Based on Figure from Clough et al. 2014

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Known Result From Geometry: Q-Criterion

- The pressure equation is given by the 2-form [Roulstone et al. 2009]

$$\alpha = \mathrm{d}q_1 \wedge \mathrm{d}q_2 - \frac{\Delta p}{2}\mathrm{d}x^1 \wedge \mathrm{d}x^2\,.$$

- Pfaffian is $f_\alpha = \frac{1}{2}\Delta p$
- Hence, the Q-criterion is recovered from the geometry:

$$\begin{aligned} \text{elliptic} \ &\Leftrightarrow f_\alpha > 0 \Leftrightarrow \Delta p > 0\,,\\ \text{hyperbolic} \ &\Leftrightarrow f_\alpha < 0 \Leftrightarrow \Delta p < 0\,,\\ \text{parabolic} \ &\Leftrightarrow f_\alpha = 0 \Leftrightarrow \Delta p = 0\,. \end{aligned}$$

Extras From Geometry: Lychagin–Rubtsov Theorem

- For $\tilde{\alpha} = \frac{1}{\sqrt{|f_\alpha|}}\alpha$, we find $\mathrm{d}\tilde{\alpha} = 0$ if and only if Δp is constant.
- Hence, by the Lychagin–Rubtsov Theorem,

$$\frac{\Delta p}{2} = (\psi_{xx}\psi_{yy} - \psi_{xy}^2)$$

  is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$ iff Δp is constant.
- So this equivalence only applies to some (relatively uninteresting) problems.

Extras From Geometry: Lychagin–Rubtsov Metric

- There is a Lychagin–Rubtsov metric on $T^*\mathbb{R}^2$ given by

$$\hat{g} = \begin{pmatrix} \frac{\Delta p}{2} I & 0 \\ 0 & I \end{pmatrix}.$$

- When pulling back to a classical solution L_ψ, we find

$$\hat{g}|_{L_\psi} = \zeta \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix}$$

where $\zeta = \Delta\psi$ is the vorticity.

Summary of Relationship

Δp	> 0	< 0	$= 0$
Dominance	Vorticity	Strain	None
$\alpha\vert_{L_\psi} = 0$	Elliptic	Hyperbolic	Parabolic
f_α	> 0	< 0	$= 0$
J^2	$-I_2$	I_2	Singular
$\hat{g}$	Riemannian $(4, 0)$	Kleinian $(2, 2)$	Degenerate*
$\hat{g}\vert_{L_\psi}$	Riemannian $(2, 0)$	Kleinian $(1, 1)$**	Degenerate*

*These degeneracies are curvature singularities.

**The $\zeta = 0$ degeneracy may occur here and be removable.

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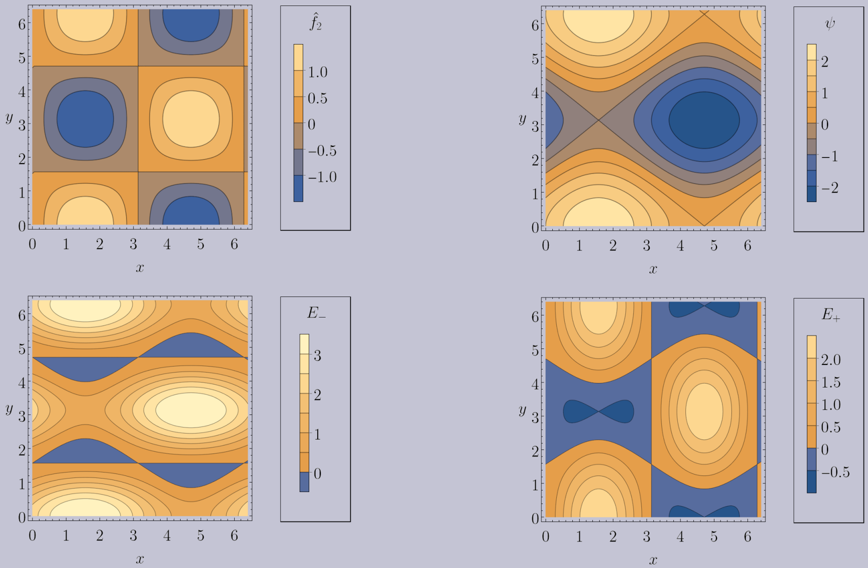
2D ABC Flow: $\psi(x,y) = \frac{3}{2}\cos(y) + \sin(x) = -\zeta$



Extension to Riemannian Manifolds

- On a Riemannian manifold (M, g), the approach is similar:

$$\Delta p + \frac{1}{2} R|v|^2 \; (+\nabla_i c^i) = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} \, .$$

- Schematically take

$$\mathrm{d}q_i \to \nabla q_i := \mathrm{d}q_i - \mathrm{d}x^j \Gamma_{ij}{}^k q_k .$$
$$I \to g .$$
$$f_\alpha = \tfrac{1}{2}\Delta p \to f_\alpha = \tfrac{1}{2}\Delta p + \tfrac{1}{4} R|v|^2 .$$

- Vorticity/strain dominance $\Leftrightarrow$ $\operatorname{sign}(f_\alpha) \Leftrightarrow \operatorname{type}(\alpha|_L = 0)$
  Pfaffian justifies Q-criterion hold on manifolds (e.g. $\mathbb{S}^2$, basin).

3. Towards Higher Monge–Ampère Equations

An Alternative Approach in 2D

- Rather than stream function ψ, work with velocity directly and consider solutions $L_v = \{(x, y, v_1(x, y), v_2(x, y))\}$.
- $\alpha|_{L_v} = 0$ gives Poission equation for pressure in terms of vorticity and strain, but now $\omega|_{L_v} = 0$ requires vanishing vorticity (bad!).
- Use a different symplectic form:

$$\begin{aligned} \varpi &= \nabla q_i \wedge \star_g \mathrm{d}x^i \\ &= \mathrm{d}q_1 \wedge \mathrm{d}x^2 - \mathrm{d}q_2 \wedge \mathrm{d}x^1 \end{aligned}$$

where $\varpi|_{L_v} = 0$ gives $\nabla_i v^i = 0$.

Higher MA Structure For Fluids

The 2-forms ϖ, α generalise in m dimensions to $(m-1)$-plectic m-forms

$$\varpi = \nabla q_i \wedge \star_g \, \mathrm{d}x^i$$
$$\alpha = \tfrac{1}{2}\nabla q_i \wedge \nabla q_j \wedge \star_g \left(\mathrm{d}x^i \wedge \mathrm{d}x^j\right) - f_\alpha \, \mathrm{vol}_M$$

With $L_v = \{(x^i, v_i(x))\}$, the equations $\varpi|_{L_v} = 0$ and $\alpha|_{L_v} = 0$ are

$$\nabla_i v^i = 0$$
$$\Delta p + R^{ij} v_i v_j = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij}$$

the divergence free equation and Poisson equation respectively.

Geometry of Higher Dimensional Fluids

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Conclusions

- (ϖ, α) not an MAS for $m > 2$, but can define and study metrics on T^*M and L_v as before.
  Signature and curvature are related to vorticity and strain.
- Can obtain topological information using Gauß–Bonnet theorem (2D) and helicity (3D)
- (ϖ, α) admits Hamiltonian reduction relating incompressible 3D flows with symmetry to compressible 2D flows with MAS [Blacker 2023].

Some Questions... (Ongoing Work)

- If not an MAS, what is (ϖ, α) for $m > 2$ and what types of equations do these correspond to?
- Can we write a Lychagin–Rubtsov theorem to classify these equations?
- LR theorem uses the structure $\alpha = J \lrcorner \varpi$ but this isn't defined for our (ϖ, α) for $m > 2$ — when is J defined and why?

Partial Higher Lychagin–Rubtsov Theorem

- An m-form ϖ on T^*M and endomorphism J on $\mathfrak{X}(T^*M)$ are <u>*compatible*</u> if, for all $k = 1, \cdots m$,

$$\varpi(JX_1, X_2, \cdots X_m) = \varpi(X_1, \cdots JX_k, \cdots X_m)$$

- For $(m-1)$-plectic form ϖ and compatible almost (para-)complex structure J, then J is integrable iff $\alpha := J \lrcorner \varpi$ is closed.

- In order to complete the theorem, we require the following:
  – Effectiveness: when does (ϖ, α) define almost (para-)complex J?
  – Pfaffian: what equations are we equivalent to and when?

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Steps Forward In Three Dimensions

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- Let $e^{ijk} = e^i \wedge e^j \wedge e^k$. Non-degenerate 3-forms in 6 dimensions (on T^*M) look like (1), (2), or (3) in some basis $\{e^i\}_{i=1}^6$ [Bryant. 2006]

$$
\begin{aligned}
&(1) \qquad e^{123} + e^{456} \\
&(2) \qquad e^{136} + e^{426} + e^{235} + e^{145} \\
&(3) \qquad e^{135} + e^{416} + e^{326}
\end{aligned}
$$

- A pair of non-degenerate 3-forms (ϖ, α) from different classes cannot define almost (para-)complex J. What about in the same class?

- Our fluid dynamical forms were not from the same class — is there an F such that $(\varpi, \alpha + F\varpi)$ are in the same class in general?

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Summary

- Introduced Monge–Ampère geometry as a tool for studying PDEs.
- Discussed application of these techniques to two-dimensional incompressible fluids, replicating and extending the Q-criterion for dominance of vorticity and strain.
- Showed how the MAS could be extended to higher dimensional flows and hinted at application of reduction to better understand topology.
- Presented initial steps to classifying higher MAS such as the Poisson and divergence-free equations in three dimensions.

Outlook

- Can we complete the higher Lychagin–Rubtsov theorem and what equations are we (locally) equivalent to?
  What are 'effectiveness' and the 'Pfaffian' in this setting?
- Is it possible to encode dynamics of fluids as well as kinematics?
  Could the vorticity equation

  $$\partial_t \zeta + \nabla(\zeta \cdot v) - \nu \Delta \zeta = 0$$

  be used as a flow equation over time t for solutions L?
- Could we consider generalised solutions of the Poisson equation?
  These represent weather fronts in [D'Onofrio et al. 2023].

Thank you!

Any questions?