Curvature Without Calculus
A Comparison Geometry Primer

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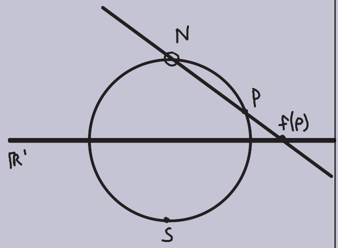
Riemannian Manifolds

- *Manifold* M looks like copies of $\mathbb{R}^n$ patched together (spheres, surfaces...)
- Equip with *Riemannian metric* g, to define inner product on each tangent space T_pM:
  - Euclidean $X \cdot Y = X^T Y$
  - In general $X \cdot Y = X^T g Y$
- View g as positive definite, symmetric matrix which varies (smoothly) with $p \in M$



The circle can be described using two copies of $\mathbb{R}$ by stereographic projection.

Intrinsic Curvature

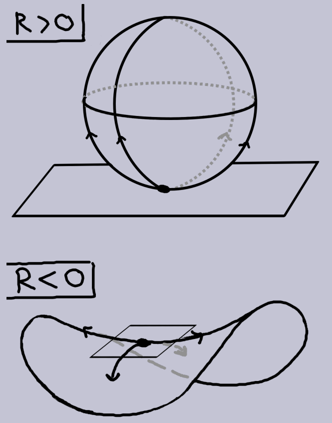
- Represents deviation of manifold from tangent space at each point (local):

  $R > 0$ – Looks like bowl.
  $R = 0$ – Looks like plane.
  $R < 0$ – Looks like saddle.

- Is given by component(s) of Riemann curvature tensor:

$$\tilde{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

so need $\mathcal{C}^2$ metric etc (or weak derivatives).



Positively and negatively curved spaces with their tangent planes.

Distance Over Derivatives

- *Metric space* is a set M with *distance* $d : M \times M \to [0, \infty)$ satisfying
  - Positive definite: $d(x, y) = 0$ iff $x = y$.
  - Symmetry: $d(x, y) = d(y, x)$
  - Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$
- Every Riemannian manifold (M, g) can be given distance

$$d_g(x, y) = \inf\left\{L_g(\gamma) \,\middle|\, \mathcal{C}^1 \text{ curves } \gamma : [a, b] \to M \text{ from } x \text{ to } y\right\},$$

where $L_g(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}\,\mathrm{d}t$ is the length of γ w.r.t. g.

Length and Distance Realisers

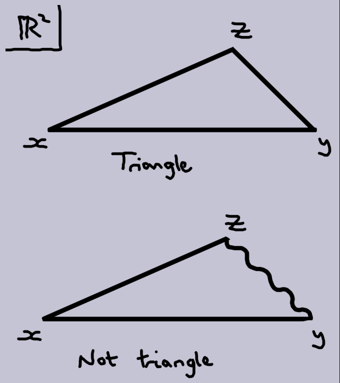
- Can measure length of curves $\gamma : [a, b] \to M$ in metric spaces using the distance d:

$$L_d(\gamma) := \sup\left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \;\middle|\; a = t_0 < \cdots < t_n = b \right\}.$$

- Call curves from x to y *distance realisers* if $L_d(\gamma) = d(x, y)$.
- Applying triangle inequality to $L_d(\gamma)$, see $L_d(\gamma) \geq d(x, y)$ for all γ; distance realisers are shortest curves.

Triangles in Metric Spaces

- *Triangles* $\Delta(x, y, z)$ are triples of points x, y, z pairwise joined by distance realisers.
- Restrict to distance realisers so side-lengths are unique for choice of vertices.
- Distance realisers (therefore triangles) between points may not exist or be unique.



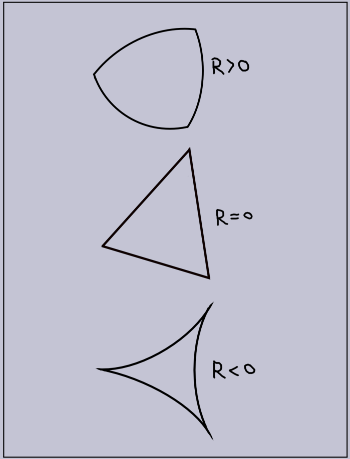
The sides of triangles in Euclidean space are straight lines.

Model Spaces of Constant Curvature

- Model spaces of constant curvature R:

$$M^2(R) := \begin{cases} \mathbb{S}^2(r) & R = \frac{1}{r^2} > 0 \\ \mathbb{R}^2 & R = 0 \\ \mathbb{H}^2(r) & R = -\frac{1}{r^2} < 0 \end{cases}.$$

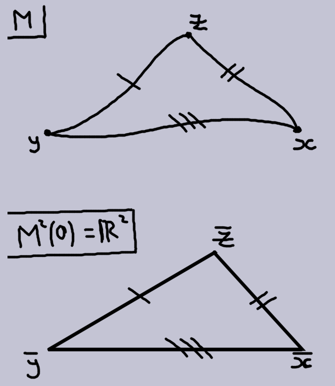
- Come equipped with natural distance (unique, complete, simply connected, 2d).
- Triangles in negatively curved spaces are "thinner" than in positively curved ones.



Comparison Triangles

- *R-Comparison triangle* for $\Delta(x, y, z) \in M$ is the unique triangle $\Delta(\bar{x}, \bar{y}, \bar{z}) \in M^2(R)$ with the same side-lengths.
- A triangle satisfies *size-bounds* for R if it has an R-comparison triangle. This requires:

$$d(x,y)+d(y,z)+d(x,z) < 2\,\mathrm{diam}(M^2(R))\,.$$

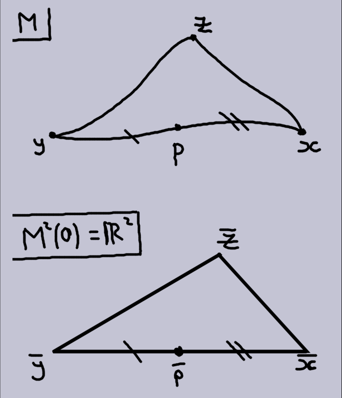


Comparison triangles tell us what our triangle would look like if it lived in model space.

Comparison Points

- Let $p \in \gamma_{xy}$ be a point on a side of $\Delta(x, y, z)$.
- The unique *R-comparison point* $\bar{p} \in \gamma_{\bar{x}\bar{y}}$ on the corresponding side of $\Delta(\bar{x}, \bar{y}, \bar{z})$ satisfies

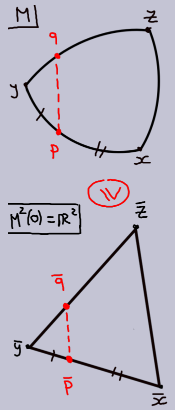
$$d(x, p) = d(\bar{x}, \bar{p}) \text{ and } d(p, y) = d(\bar{p}, \bar{y})\,.$$



Moving along $\gamma_{\bar{x}\bar{y}}$ by $d(x, p)$ gives us a point $\bar{p} \in M^2(0)$ corresponding to p.

Curvature Bounds

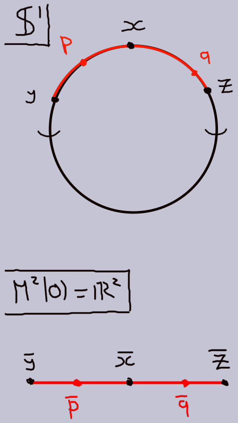
- *$(\geq R)$-comparison nbhd* is open $U \subseteq M$:
  - all distance realisers exist in U.
  - for all $\Delta(x, y, z) \subset U$ and $p, q \in \Delta(x, y, z)$, the comparison points satisfy $d(p, q) \geq d(\bar{p}, \bar{q})$.
- Metric spaces M have curvature:
  - *bounded below by R* if c. nbhds cover M
  - *globally bounded below by R* if $U = M$.
- Use $\leq$ for curvature *bounded above*



Comparing distance between $p, q \in M$ and their comparison points in $\mathbb{R}^2$. M has $\mathrm{CBB}(0)$.

Example: Circle (Part 1)

- Unit circle $\mathbb{S} \subset \mathbb{R}^2$ with $d(x, y) =$ Euclidean length of shortest arc.
- Cover circle with open intervals of length $< \pi$ — triangles look like image.
- 0-Comparison triangle degenerates to segment, so $d(p, q) = d(\bar{p}, \bar{q})$.
- $\mathbb{S}$ has curvature bounded above and below by 0, so has curvature 0 locally.



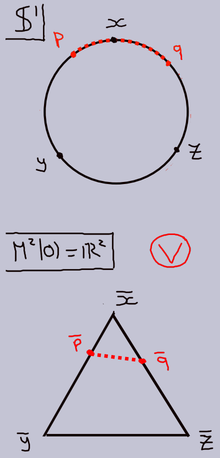
Small triangles in $\mathbb{S}^1$ look like straight lines in $\mathbb{R}^2$.

Example: Circle (Part 2)

- Globally, $\mathbb{S}$ only has curvature bounded below by 0 and not above.
- Consider a triangle which covers the circle as in image, then

$$
\begin{aligned}
d(p,q) &= d(p,x) + d(x,q) \quad \text{(distance realiser)} \\
&= d(\bar{p},\bar{x}) + d(\bar{x},\bar{q}) \quad \text{(comp. points)} \\
&> d(\bar{p},\bar{q}) \quad \text{(strict triangle inequality)}
\end{aligned}
$$

- Has curvature 1 (check by comparing triangles to $\mathbb{S}^2$).

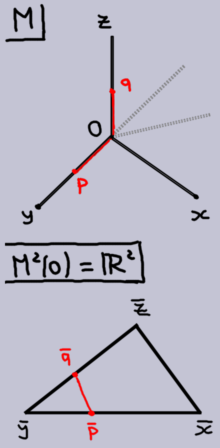


Large triangles in $\mathbb{S}^1$ are wider than in $\mathbb{R}^2$.

Example: Finite Connected Metric Graphs

- Union of distance realising segments.
- Locally look like a segment or a fan, so curvature bounded above by 0.
- For neighbourhood containing only a segment, $d(p,q) = d(\bar{p},\bar{q})$ trivial, for fan see diagram:

$$\begin{aligned} d(p,q) &= d(p,z) - d(q,z) \quad \text{(segments)} \\ &= d(\bar{p},\bar{z}) - d(\bar{q},\bar{z}) \quad \text{(comp. points)} \\ &\leq d(\bar{p},\bar{q}) \quad \text{(triangle inequalty)} \end{aligned}$$



Small triangles in graphs look like straight lines or tripods.

Some Fun Results From Literature

- Complete, "intrinsic" metric spaces have global curvature bounded below by R iff they have curvature bounded below by R [Perelman–Burago–Gromov].
- Complete, intrinsic metric spaces with curvature bounded below by $R > 0$ have diameter $\leq \frac{\pi}{\sqrt{R}}$ [Bonnet–Myers].
- Limit of metric spaces with curvature bounded below by R is a metric space with curvature bounded below by R [Gromov–Hausdorff].
- Every compact, intrinsic metric space is the limit of finite, connected graphs [Gromov–Hausdorff].

The Link To My Work

- This talk addressed the low regularity (no calculus) version of Riemannian manifolds – what about Lorentzian?
- Einstein equations admit low regularity Lorentzian metrics as solutions

$$R_{\mu\nu} + (\Lambda - \frac{1}{2} R g_{\mu\nu}) = C T_{\mu\nu}$$

  Occur in models of grav. waves, cosmic strings, certain singularities.
- *Lorentzian pre-length spaces* are analogues of metric spaces in this setting (work similarly, but only measure time separation...)

Thank you! Any questions?