Semester 2, 2020/21
University of Surrey

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Numerical and Computational Methods

Exercise sheet 1

Exercise 1.1:

Write a function that computes the cube root of a using the second algorithm:

$$x_{n+1} = x_n - \frac{x_n^3 - a}{3x_n^2} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right), \quad n = 0, 1, 2, \dots$$

How many iterations are required to obtain an accuracy of $\left|x^3 - a\right| < 10^{-6}$. Choose the initial condition $x_0 = a$ and apply the algorithm to

(i) $a = 3.375$,

(ii) $a = -8$,

(iii) $a = 1331$.

Exercise 1.2:

Implement a function that computes the 2-norm of a vector $x \in \mathbb{R}^n$, i.e.,

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2}.$$

Exercise 1.3:

Define $x^{(0)} = [1, 1, 1]^\top$ and

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Implement the following algorithm:

(i) $\widetilde{x}^{(n+1)} = A x^{(n)}$.

(ii) $x^{(n+1)} = \dfrac{1}{\left\|\widetilde{x}^{(n)}\right\|} \widetilde{x}^{(n)}$.

(iii) $\alpha^{(n+1)} = \dfrac{x^{(n+1)^\top} A x^{(n+1)}}{x^{(n+1)^\top} x^{(n+1)}}$.

Carry out the first ten iterations. What does this algorithm compute?

Exercise 1.4:

Implement the Laplace expansion to compute the determinant of a matrix $A \in \mathbb{R}^{n\times n}$. Measure the runtimes (use `tic` and `toc`) for different n and compare these results with Matlab's `det` function.

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Numerical and Computational Methods

Exercise sheet 2

Exercise 2.1:

Write a Matlab function `luSolver` that solves systems of linear equations. You can use the functions

- `luDecomposition`,
- `forwardSubstitution`,
- `backwardSubstitution`

from the lecture notes.

Exercise 2.2:

Let $H \in \mathbb{R}^{n \times n}$ be the Hilbert matrix of size n. Define $b \in \mathbb{R}^n$ by

$$b_i = \sum_{j=1}^{n} \frac{1}{i+j-1}.$$

(i) Solve the system of linear equations $H\,x = b$.

(ii) Define $\widetilde{b} = b + \varepsilon\, e_1$, where $e_1 = [1, 0, \ldots, 0]^\top$. Choose $\varepsilon = 10^{-3}$.

(iii) Solve the perturbed system of equations $H\,x = \widetilde{b}$.

(iv) Explain the results.

(v) Plot the runtime of the function `luSolver` for different n.

Exercise 2.3:

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. Show that:

(i) All eigenvalues of A are positive.

(ii) $\det(A) > 0$.

(iii) A is invertible.

(iv) A^{-1} is symmetric and positive definite.

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Numerical and Computational Methods

Exercise sheet 3

Exercise 3.1:

Given the interval $I = [1.75, 2]$, consider the fixed-point iteration $x_{k+1} = F(x_k)$, with $F(x) = 1 + \frac{1}{x} + \frac{1}{x^2}$.

(i) Show that F is a contraction.

(ii) Carry out the first 20 iterations using $x_0 = 1.8$. Analyze the convergence of the fixed-point iteration.

(iii) The root of which polynomial is computed here?

Hint: Let F be a differentiable function defined on an interval $I \subset \mathbb{R}$. Suppose that $|F'(x)| < M$ for all $x \in I$. Using the mean-value theorem, it holds that

$$F(x) - F(y) = F'(\xi)(x - y)$$

for some ξ between x and y. If follows that $|F(x) - F(y)| \leq M\,|x - y|$. That is, M is a Lipschitz constant.

Exercise 3.2:

Apply the bisection method to $f(x) = x - 1 - \frac{1}{x} - \frac{1}{x^2}$. Use the initial interval $I = [1.75, 2]$. Carry out the first ten iterations. In which interval is the root of f?

Exercise 3.3:

Given the data points

x_i	-2	0	3
y_i	5	-1	8

,

determine the Lagrange polynomials L_i and the interpolating polynomial p with $p(x_i) = y_i$. Plot your solution in Matlab and verify the results.

Exercise 3.4:

Use Newton interpolation for the data from Exercise 3.3.

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Numerical and Computational Methods

Exercise sheet 4

Exercise 4.1:

Find the polynomial p of order 3 with $p(1) = 10$, $p(2) = 26$, $p'(2) = 23$, and $p''(2) = 16$.

Exercise 4.2:

Given the interval $I = [-1, 1]$ and the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, \mathrm{d}x,$$

approximate the function $f(x) = \exp(x)$ using Gauss approximation. Choose the basis functions $\{1, x, x^2\}$ and solve the resulting system of linear equations. Compute the integrals with the aid of the function `quad`. Plot the results.

Exercise 4.3:

Implement the composite Simpson's rule in Matlab. Compute

$$\int_{5}^{10} \frac{1}{x}\, \mathrm{d}x$$

for different n and compare the results with the analytical solution.

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Exercise sheet 5

Exercise 5.1:

Implement the standard fourth-order Runge–Kutta method given by the following Butcher tableau:

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

Solve the initial value problems

(i) $\dot{x} = -2tx^2, \quad x(0) = 1,$

(ii) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Compare the solutions with the analytical solutions.

Exercise 5.2:

Show that applying the aforementioned Runge–Kutta method to the ODE $\dot{x} = Ax$, where $A \in \mathbb{R}^{n \times n}$, results in a method of the form $x_{k+1} = Bx_k$. Compute the matrix B. Interpret the results.

Exercise 5.3:

Use the results from Exercise 5.2 to estimate for which $z = h\lambda$ the method is stable. Plot the stability region.