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Alexandrov's Patchwork and the Bonnet–Myers Theorem for Lorentzian length spaces

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Abstract

We present key initial results in the study of global timelike curvature bounds within the Lorentzian pre-length space framework. Most notably, we construct a Lorentzian analogue to Alexandrov's Patchwork, thus proving that suitably nice Lorentzian pre-length spaces with local upper timelike curvature bound also satisfy a corresponding global upper bound. Additionally, for spaces with global lower bound on their timelike curvature, we provide a Bonnet–Myers style result, constraining their finite diameter. Throughout, we make the natural comparisons to the metric case, concluding with a discussion of potential applications and ongoing work.

Keywords: Lorentzian length spaces, synthetic curvature bounds, globalization, triangle comparison, metric geometry, Lorentzian geometry, hyperbolic angles

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Contents

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1 Introduction

By utilising the theory of metric length spaces, the scope of many results in differential geometry can be extended beyond the setting of smooth manifolds. In particular, metric length spaces are a key tool in the abstraction of the fundamental properties of Riemannian manifolds, to structures of lower regularity [BBI01, BH99]. In this so-called synthetic approach, curvature bounds (locally/in the small) are constructed via the comparison properties of geodesic triangles and are used to tame some of the more pathological behaviour of such length spaces. A semi-Riemannian extension of these comparison methods has also been developed in [AB08, Har82].

The properties that arise from supplementing length spaces with global curvature bounds (producing spaces of Alexandrov or CAT(k) type) have also been significant for the application of metric length spaces to problems in a variety of fields, from dynamical systems to group theory. An illustrative result in the case of curvature bounded below is given by the stability, under Gromov–Hausdorff limits, of global lower curvature bounds on metric spaces [BGP92, Kap02]. We can see the impact of spaces with global upper curvature bounds by looking to algebraic topology: the fundamental group of any complete metric space with curvature bounded above by zero has no non-trivial finite subgroups [BBI01, Corollary 9.3.2].

As such, a salient question asks under which conditions can curvature bounds, which are imposed locally, be extended to hold in the large? For spaces with curvature bounded above, Alexandrov [Ale57] demonstrated that, provided we also have unique geodesics which vary continuously with their endpoints, local CAT(k) spaces are CAT(k). Extending the work of Toponogov [Top59] on Riemannian manifolds, it was shown by Perelman [BGP92] that for curvature bounded below, complete length spaces are sufficient. In fact, a generalization of the Bonnet–Myers Theorem follows as a natural corollary of the aforementioned result [BBI01], bounding the diameter of complete length spaces with local curvature bounded below. A further globalization result appears in [Pet16], where the author treats completions of geodesic spaces with curvature bounded below.

Analysis of low regularity Lorentzian metrics has become increasingly pertinent in the study of general relativity and physically relevant spacetimes, which may feature cosmic strings and gravitational waves, see for example [Ren05, Vic90, PSSu16]. Hence, the introduction of the Lorentzian pre-length space [KS18] as an extension of the basic objects described by causal spaces [KP67], has led to the rapid development of a synthetic Lorentzian framework, mirroring the growth of the theory of metric length spaces several decades ago [AGKS21, GKS19, KS22]. In particular, [KS18] develop comparison methods for synthetic Lorentzian geometry, in order to impose curvature bounds on Lorentzian pre-length spaces. Equivalent approaches have also been proposed by [BS22, BMS22], however none of these works relate curvature bounds enforced globally to those imposed on neighbourhoods. Furthermore, while the development of metric length spaces was guided by disciplines such as group theory and the study of partial differential equations, alongside its purely geometric origin, research into Lorentzian pre-length spaces has, for the most part, focused on their apparent necessity in general relativity.

This paper, continuing from the work of [BR22, Rot22, BS22], aims to develop suitable Lorentzian analogues to some of the fundamental results of metric length spaces, in order to facilitate the wider application of the Lorentzian length space framework. In particular, given their myriad of applications in the metric setting, we focus on the notion of spaces with global timelike curvature bounds.

An outline of the paper is as follows. We begin in Section 2 by re-iterating some basic definitions regarding Lorentzian pre-length spaces, τ-length, and the causal ladder. We also introduce the notion of a regular Lorentzian pre-length space, the technique of triangle comparison, and both local and global timelike curvature bounds. Similar definitions in the context of metric spaces are also provided for convenience and comparison. In Section 3, we provide a summary of some globalization results from the metric setting that we wish to mirror with our 'Lorentzified' constructions. In particular, we give explicit statements of the so-called Alexandrov's Patchwork, Toponogov's Theorem, and the Bonnet–Myers Theorem. The main results of this paper are proven within Section 4, which is split into two parts. The first part concerns globalization of upper timelike curvature bounds via gluing of timelike triangles and culminates in the following theorem:

Theorem 4.6 (Alexandrov's Patchwork Globalization, Lorentzian version)**.** *Let X be a strongly causal, non-timelike locally isolating, and regular Lorentzian pre-length space which has (local) timelike curvature bounded above by $K \in \mathbb{R}$. Suppose that X satisfies* (i) *and* (ii) *in Definition 2.7, i.e., $\tau^{-1}([0, D_K))$ is open and τ is continuous on that set, and for all $x \ll y$ in X with $\tau(x, y) < D_K$, there exists a geodesic joining them. Additionally assume that the geodesics between timelike related points with τ-distance less than D_K are unique. Let G be the geodesic map of X restricted to the set $\{(x, y, t) \in \ll \times [0, 1] \,|\, \tau(x, y) < D_K\} = \tau^{-1}((0, D_K)) \times [0, 1]$ and assume that G is continuous. Then X also satisfies Definition 2.7.*(iii)*, in particular it is a $\leq K$-comparison neighbourhood and X has global curvature bounded above by K.*

The second part concerns a result akin to the Bonnet–Myers Theorem, bounding the finite timelike diameter of Lorentzian pre-length spaces, using global lower timelike curvature bounds. More precisely:

Theorem 4.11 (Bound on the finite diameter)**.** *Let X be a strongly causal, locally causally closed, regular, and geodesic Lorentzian pre-length space which has global curvature bounded below by K. Assume $K < 0$. Assume that X possesses the following non-degeneracy condition: for each pair of points $x \ll z$ in X we find $y \in X$ such that $\Delta(x, y, z)$ is a non-degenerate timelike triangle. Then* $\mathrm{diam}_{\mathrm{fin}}(X) \leq D_K$.

We conclude the paper with a discussion of ongoing research into the globalization of timelike curvature bounded below, as well as highlighting a potential application of Lorentzian pre-length spaces and global curvature bounds to the theory of causal sets.

2 Preliminaries

In this section we collect basic results from the theory of Lorentzian length spaces that will be of use in this article. For more details, we refer the interested reader to [KS18]. We also recall the corresponding elementary concepts from metric geometry, as a showcase of the tools used in the globalization of metric curvature bounds. For details regarding their precise application, see [BH99, BBI01].

2.1 Introduction to Lorentzian pre-length spaces

Let us begin by summarising the fundamentals of the Lorentzian length space framework, pioneered by Kunzinger and Sämann in [KS18]. In particular, we present rungs from the causal ladder [ACS20, Rot22] which will be necessary in later proofs, in addition to describing the use of triangle comparison to test for curvature bounds. First, let us define a Lorentzian pre-length space:

Definition 2.1 (Lorentzian pre-length space)**.** *Let (X, d) be a metric space, $\ll, \leq$ two relations on X, and $\tau : X \times X \to [0, \infty]$ a function. The quintuple $(X, d, \ll, \leq, \tau)$ is then called a* Lorentzian pre-length space *if it satisfies the following:*

(i) $(X, \ll, \leq)$ *is a* causal space*, i.e., $\leq$ is a reflexive and transitive relation and $\ll$ is a transitive relation contained in $\leq$.*

(ii) *τ is lower semi-continuous with respect to d.*

(iii) $\tau(x, z) \geq \tau(x, y) + \tau(y, z)$ *for $x \leq y \leq z$ and* $\tau(x, y) > 0 \iff x \ll y$.

In this case, τ is called the time separation function*, with $\ll$ and $\leq$ referred to as the* timelike *and* causal relations*, respectively. All of these concepts are motivated by the corresponding notions in spacetimes.*

A Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ will usually be denoted simply by X, where the latter is clear. When referring to causal (or timelike) pasts and futures, we shall use the standard notation, e.g., $I^+(x) := \{y \in X \mid x \ll y\}$ and $J^-(x) := \{y \in X \mid y \leq x\}$. In particular, for diamonds, we shall use the notation $J(x, z) := J^+(x) \cap J^-(z) = \{y \in X \mid x \leq y \leq z\}$.

Definition 2.2 (Causal and timelike curves)**.** *Let X be a Lorentzian pre-length space. A locally Lipschitz curve $\gamma : [a, b] \to X$ is called* future-directed causal *(respectively* timelike*), if $\gamma(s) \leq \gamma(t)$ (respectively $\gamma(s) \ll \gamma(t)$) for all $s < t$ in $[a, b]$. A* past-directed *curve is defined analogously with the relations in X reversed. To make our terminology less cumbersome and avoid repeated reference to time orientation, we assume causal curves are future-directed, unless it is explicitly stated otherwise.*

Definition 2.3 (τ-length and geodesics)**.** *Let $\gamma : [a, b] \to X$ be a causal curve from x to y in a Lorentzian pre-length space X.*

*(i) We define its τ-*length *as*

$$L_\tau(\gamma) := \inf\left\{ \sum_{i=0}^{n-1} \tau(\gamma(t_i), \gamma(t_{i+1})) \,\middle|\, a = t_0 < t_1 < \ldots < t_n = b, n \in \mathbb{N} \right\}. \tag{2.1}$$

(ii) By definition we always have $L_\tau(\gamma) \leq \tau(x, y)$. In the case of equality, we call γ a distance-realizer *or* geodesic*.*

(iii) X is called geodesic *if, for each pair of causally related points, there exists a geodesic connecting them. X is called* uniquely geodesic *if it is geodesic and the geodesic between each pair of points is unique. If a geodesic γ between $x \ll y$ is unique, or if it is not unique and a choice of geodesic has either been made or is inconsequential, then we denote the geodesic by γ_{xy}. Unless otherwise mentioned, we shall assume γ_{xy} is parameterized by $[0, 1]$ and has constant speed, i.e., $\tau(\gamma(s), \gamma(t)) = \tau(x, y)|t - s|$ for all $s < t \in [0, 1]$.*

The additional assumptions given above, that geodesics are parameterized by $[0, 1]$ and have constant speed, pose no technical issues when dealing with geodesics (between timelike related points) which are timelike. In formulating the main results of this paper, we consider Lorentzian pre-length spaces which are well-behaved, in the sense that geodesics between timelike related points are always timelike. In order to encode this as a property of the space, we now introduce the notion of regularity.

Technically, this is very similar to the definition of a regularly localizable Lorentzian length space, cf. [KS18, Definition 3.16]. As we do not require much of the additional structure provided by Lorentzian length spaces, we prefer to formulate everything in terms of Lorentzian pre-length spaces.

Definition 2.4 (Regular Lorentzian pre-length space). *A Lorentzian pre-length space X is called* regular *if for all $x, y \in X$ such that $x \ll y$ all geodesics connecting x and y are timelike.*

In general, a geodesic between timelike related points may contain a null piece, see for example the causal funnel [KS18, Example 3.19].

Before providing a definition of timelike curvature bounds, it is now necessary to introduce the notion of triangle comparison:

Definition 2.5 (Model spaces and triangle comparison). *Let X be a Lorentzian pre-length space. We define the following:*

(i) A timelike triangle $\Delta(x, y, z)$ *in X is a collection of three timelike related points $x \ll y \ll z$ and three pairwise connecting geodesics γ_{xy}, γ_{yz} and γ_{xz} between them. To indicate a point p lies on the triangle, we write $p \in \Delta(x, y, z)$, with $p \in \gamma_{xy}$ used to specify which side.*

(ii) By $\mathbb{L}^2(K)$ we denote the Lorentzian model space of constant sectional curvature K. That is, $\mathbb{L}^2(0)$ is the Minkowski plane and $\mathbb{L}^2(K)$, for $K > 0$ and $K < 0$, is an appropriately scaled version of 2-dimensional deSitter or anti-deSitter spacetime, respectively, see [KS18, Definition 4.5].

(iii) Given a timelike triangle $\Delta(x, y, z)$ in X, we call a timelike triangle $\Delta(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{L}^2(K)$ whose sides have the same sidelengths a comparison triangle *for $\Delta(x, y, z)$. We assume that all timelike triangles in the remainder of the paper satisfy appropriate size bounds,*[1] *unless it is explicitly stated otherwise.*

(iv) Let $p \in \gamma_{xy}$ (analogously for γ_{xz} and γ_{yz}) be a point on some side of the triangle $\Delta(x, y, z)$ in X and let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle for $\Delta(x, y, z)$. The comparison point *for p, in $\Delta(\bar{x}, \bar{y}, \bar{z})$, is the (unique) point $\bar{p} \in \gamma_{\bar{x}\bar{y}}$, whose τ-distance to the endpoints of $\gamma_{\bar{x}\bar{y}}$ is the same as the τ-distance from p to the respective endpoints of γ_{xy}.*

(v) We call a timelike triangle $\Delta(x, y, z)$ non-degenerate *if the inequality $\tau(x, z) > \tau(x, y) + \tau(y, z)$ holds. In this case, any associated comparison triangle is non-degenerate in the visual sense, i.e. not just a geodesic segment.*

[1] This is an equivalent condition for the existence of a comparison triangle in a chosen model space, see Definition 2.6.

Before finally providing the definition of timelike curvature bounds, we raise the following technical detail. We feel that in general, but especially in the context of this paper, the original definition of timelike curvature bounds given in [KS18] should be modified slightly. More precisely, the defining properties of a comparison neighbourhood (see [KS18, Definition 4.7]) should only hold wherever τ is "not too large" for the comparison space. This is mainly as a result of exotic behaviour exhibited by anti de-Sitter space (AdS), the model space for constant negative timelike curvature. In AdS, geodesics (in the smooth sense) stop being maximizing when they exceed length π (when $K = -1$, otherwise this bound is appropriately scaled).

Visually, this can be explained as follows: place two points x and y on the "equator" of AdS such that they are not antipodal and connect them via the longer geodesic (recall that geodesics in AdS arise by intersecting the space with a plane through 0 and the two endpoints). Then we can create curves of increasing lengths by going out to infinity, say to the right of the equator. In particular, there is no longest curve joining x and y and $\tau(x, y) = \infty$. This is related to AdS not being globally hyperbolic; indeed, the maximal globally hyperbolic subset of AdS has (ordinary) diameter π. This is clearly pathological behaviour exclusive to the Lorentzian case, however there is some similarity with metric model spaces of positive curvature, namely that between points which are exactly a distance π apart there exist infinitely many geodesics (compare the antipodal points on AdS with those on the sphere).

In order to provide our updated definition, we have to introduce the so-called finite diameter of a space. This is essentially the diameter, i.e., the supremum of all values of τ, but we explicitly exclude ∞ as a value because of the nature of AdS. Note that, despite the nomenclature, the finite diameter of a Lorentzian pre-length space need not be finite.

Definition 2.6. *Let X be a Lorentzian pre-length space.*

(i) The finite diameter *of X is*

$$\mathrm{diam}_{\mathrm{fin}}(X) = \sup(\{\tau(x, y) : x \ll y\} \setminus \{\infty\}), \tag{2.2}$$

i.e., the supremum of all values τ takes except ∞.

(ii) By D_K we denote the finite diameter of $\mathbb{L}^2(K)$. In particular,

$$D_K = \mathrm{diam}_{\mathrm{fin}}(\mathbb{L}^2(K)) = \begin{cases} \infty, & \text{if } K \geq 0, \\ \frac{\pi}{\sqrt{-K}}, & \text{if } K < 0. \end{cases} \tag{2.3}$$

Note that a triangle $\Delta(x, y, z)$ satisfies size-bounds for $\mathbb{L}^2(K)$ precisely if $\tau(x, z) < D_K$, cf. [KS18, Lemma 4.6].

Let us now make our point concrete: *all* properties of a comparison neighbourhood should respect the appropriate range of values of τ. In particular, we do not care whether τ is continuous near points separated by a distance which cannot be realized in the model space, and we should also not require such points to possess a joining geodesic. On the one hand, this refined definition is somehow more in alignment with its metric counterpart. For example, in the definition of $\mathrm{CAT}(k)$ spaces, cf. [BH99, Definition II.1.1], the authors explicitly only require that there exists a geodesic between points which are less than the diameter of the corresponding model space apart. On the other hand, curvature bounds should morally not be concerned with behaviour which cannot be realized in the model space.

Definition 2.7 (Timelike curvature bounds). *Let X be a Lorentzian pre-length space. An open subset U is called a* timelike $\geq K$-comparison neighbourhood *(or* timelike $\leq K$-comparison neighbourhood*) if:*

(i) τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$ and $(U \times U) \cap \tau^{-1}([0, D_K))$ is open.

(ii) For all $x, y \in U$ with $x \ll y$ and $\tau(x, y) < D_K$ there exists a geodesic connecting them which is contained entirely in U.

(iii) Let $\Delta(x, y, z)$ be a timelike triangle in U, with p, q two points on the sides of $\Delta(x, y, z)$. Let $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle in $\mathbb{L}^2(K)$ for $\Delta(x, y, z)$ and $\bar{p}, \bar{q}$ comparison points for p and q, respectively. Then

$$\tau(p, q) \leq \tau(\bar{p}, \bar{q}) \qquad (\text{or } \tau(p, q) \geq \tau(\bar{p}, \bar{q})). \tag{2.4}$$

We say X has timelike curvature bounded below by K *if it is covered by timelike $\geq K$-comparison neighbourhoods. Likewise, X has* timelike curvature bounded above by K *if it is covered by timelike $\leq K$-comparison neighbourhoods.*

We say X has global timelike curvature bounded below by K *if X itself is a $\geq K$-comparison neighbourhood. Similarly, X has* global timelike curvature bounded above by K *if X is a $\leq K$-comparison neighbourhood.*

Note that within a $\geq K$ comparison neighbourhood, $p \ll q$ implies $\bar{p} \ll \bar{q}$, and within a $\leq K$ comparison neighbourhood, $\bar{p} \ll \bar{q}$ implies $p \ll q$.

Remark 2.8 (Global curvature bound of AdS). Note that with the above definition, $\mathbb{L}^2(-1)$ satisfies global curvature bounds both above and below. In contrast, $\mathbb{L}^2(-1)$ does not satisfy a global curvature bound with respect to the original definition of [KS18], since it does not satisfy their conditions for a comparison neighbourhood: τ is neither finite nor continuous, and for x, y with $\tau(x, y) > \pi$, there is no geodesic joining them. $\triangle$

When treating local curvature bounds, we consider spaces covered by comparison neighbourhoods. In establishing a globalization theorem, a precise description of the aforementioned covering will be useful. To this end, one step of the causal ladder, namely strong causality, is crucial:

Definition 2.9 (Strong causality)**.** *A Lorentzian pre-length space X is called* strongly causal *if $\mathcal{I} := \{I(x,y) \,|\, x, y \in X\}$ is a subbasis for the topology induced by d.*

It turns out, however, that a finite intersection of diamonds inside an arbitrary neighbourhood is not sufficient; we will actually require the existence of a timelike diamond inside any neighbourhood, i.e., $\mathcal{I}$ must be a basis for the topology. This is possible under the additional assumption of non-timelike local isolation:

Definition 2.10 (Non-timelike local isolation)**.** *A Lorentzian pre-length space X is said to be* non-timelike locally isolating *if for all $x \in X$ and all neighbourhoods U of x in X there exist $x_-, x_+ \in U$ such that $x_- \ll x \ll x_+$.*

Proposition 2.11 (Diamonds form basis)**.** *Let X be a strongly causal and non-timelike locally isolating Lorentzian pre-length space. Then $\mathcal{I}$ forms a basis for the topology. In particular, given any neighbourhood of any point, we can construct a timelike diamond containing the point, such that the diamond and its governing points are also contained in the neighbourhood.*

Proof. See [Rot22, Lemma 3.5]. □

Conveniently, the previous proposition also proves to be the perfect tool for highlighting the relationship between comparison neighbourhoods in the sense of [KS18, Definition 4.7] and the modification we propose in Definition 2.7. Indeed, under assumptions of strong causality and non-timelike local isolation (as in AdS, for example), the two definitions may be used interchangeably and one may construct comparison neighbourhoods in the sense of [KS18] via the following lemma:

Lemma 2.12 (Automatic size bounds)**.** *Let X be a strongly causal and non-timelike locally isolating Lorentzian pre-length space which has timelike curvature bounded below (above) by some $K \in \mathbb{R}$. Then X is covered by timelike $\geq K$-comparison (resp. $\leq K$-comparison) neighbourhoods U where $\tau|_{U \times U} < D_K$. In particular, these U are curvature comparison neighbourhoods in the sense of the old definition [KS18, Definition 4.7] and all timelike triangles are realizable.*

Moreover, all curvature comparison neighbourhoods in the sense of [KS18, Definition 4.7] are $\leq K$-comparison neighbourhoods in the sense of Definition 2.7. In particular, a Lorentzian pre-length space has (local) curvature bounds in the sense of 2.7 if and only if it has curvature bounds in the sense of [KS18, Definition 4.7].

Proof. Let $x \in X$ and $\tilde{U}$ be a $\geq K$-comparison (resp. $\leq K$-comparison) neighbourhood of x. We have that $\tau(x,x) = 0$. $(\tilde{U} \times \tilde{U}) \cap \tau^{-1}([0, D_K))$ is open and contains (x,x), thus we find a small neighbourhood V of x such that $V \times V \subseteq (\tilde{U} \times \tilde{U}) \cap \tau^{-1}([0, D_K))$. By Proposition 2.11, we find $x_-, x_+ \in V$ such that $x_- \ll x \ll x_+$ and $x \in I(x_-, x_+) \subset V$, and set $U = I(x_-, x_+)$. It follows that $\tau(x_-, x_+) < D_k$, and hence $\tau|_{U\times U} < D_k$.

We now verify that U is a $\geq K$-comparison (resp. $\leq K$-comparison) neighbourhood in the sense of [KS18, Definition 4.7]: Clearly τ is finite and continuous on $U \times U$. Furthermore, by causal convexity[2] of timelike diamonds, geodesics between points in U remain in U. Finally, U inherits property *(iii)* of Definition 2.7 from the comparison neighbourhood $\tilde{U}$.

If τ is continuous on $U \times U$ it is also continuous on $(\tilde{U} \times \tilde{U}) \cap \tau^{-1}([0, D_K))$, and this set is open by the (local) continuity of τ. □

In metric geometry, there are several reformulations of curvature bounds expressed via classical triangle comparison, using angles, for example. Alternative versions also exist for timelike curvature bounds (see [BS22, BMS22]) and several of these characterizations will prove useful in our context. Before we state these explicitly, let us introduce some more terminology:

Definition 2.13 (K-comparison angles and sign)**.** *Let X be a Lorentzian pre-length space, $K \in \mathbb{R}$, $\Delta(x,y,z)$ a timelike triangle in X, and $\Delta(\bar{x}, \bar{y}, \bar{z})$ a comparison triangle in $\mathbb{L}^2(K)$ for $\Delta(x,y,z)$.*

(i) The K-comparison angle *at x is defined as the ordinary hyperbolic angle at $\bar{x}$ between $\bar{y}$ and $\bar{z}$:*

$$\tilde{\measuredangle}_x^K(y,z) := \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y}, \bar{z}) = \operatorname{arcosh}(|\langle \gamma'_{\bar{x}\bar{y}}(0), \gamma'_{\bar{x}\bar{z}}(0)\rangle|), \tag{2.5}$$

where we assume the mentioned geodesics to be unit speed parameterized.

(ii) The sign σ *of a K-comparison angle is the sign of the corresponding inner product (in the $-, +, \cdots, +$ convention). That is, in this notation, the sign is -1 if the angle is measured at x or z and 1 if the angle is measured at y.*

(iii) The signed K-comparison angle *is defined as $\tilde{\measuredangle}_x^{K,S}(y,z) := \sigma\tilde{\measuredangle}_x^K(y,z)$.*

Definition 2.14 (Angles and hinges)**.** *Let X be a Lorentzian pre-length space and let α and β be two timelike curves of arbitrary time orientation emanating at $\alpha(0) = \beta(0) =: x$.*

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[2]Recall that a subset A of a Lorentzian pre-length space X is called *causally convex* if for all $p, q \in A$ it holds that $J(p,q) \subseteq A$. Causal and timelike diamonds are among the most prominent examples of causally convex sets.

(i) The angle *between α and β, if it exists, is defined as*

$$\measuredangle_x(\alpha,\beta) \coloneqq \lim_{s,t\to 0} \tilde{\measuredangle}^0_x(\alpha(s),\beta(t)), \tag{2.6}$$

where the limit only considers values of s and t for which the triple $(x,\alpha(s),\beta(t))$ (or some permutation thereof) forms a timelike triangle.

(ii) The sign *σ of an angle is -1 if α and β have the same time orientation and 1 otherwise.*

(iii) An angle at a point $x\in X$ and the associated geodesics are called a hinge*, which we will denote by (α,β).*

(iv) Given $K\in\mathbb{R}$ and a hinge (α,β) at x in X, we call a hinge $(\bar\alpha,\bar\beta)$ at $\bar x$ in $\mathbb{L}^2(K)$ whose corresponding sides have the same lengths and time orientations and satisfy $\measuredangle_x(\alpha,\beta)=\measuredangle^{\mathbb{L}^2(K)}_{\bar x}(\bar\alpha,\bar\beta)$ a K-comparison hinge for (α,β).

As we will never work with different model spaces simultaneously and as the limit in (2.6), cf. [BS22, Proposition 2.14] is the same regardless of the model space in which it is considered, we will usually drop the superscript in the comparison angle and just write $\tilde{\measuredangle}_x(y,z)$.

Now we highlight some alternative formulations of curvature bounds, beginning with monotonicity comparison. The remaining results in this subsection are also valid when considering upper curvature bounds, where inequalities are switched in the obvious way. However, as we will only need our reformulations in the setting of lower curvature bounds, we shall not state the former case herein.

As provided in [BS22, Definition 4.8], monotonicity comparison (specifically point *(ii)*) requires the additional technical assumption that the neighbourhoods considered are *strictly timelike geodesic*, meaning that for any two close enough timelike related points there is a *timelike* geodesic joining them. This can be omitted by instead assuming that our Lorentzian pre-length space is regular as in Definition 2.4; monotonicity comparison then takes the following form:

Definition 2.15 (K-Monotonicity comparison)**.** *Let $K\in\mathbb{R}$ and let X be a regular Lorentzian pre-length space. X is said to satisfy* K-monotonicity comparison from below *if every point in X possesses an open neighbourhood U such that:*

(i) τ is continuous on $(U\times U)\cap\tau^{-1}([0,D_K))$ and $(U\times U)\cap\tau^{-1}([0,D_K))$ is open.

(ii) *For all $x, y \in U$ with $x \ll y$ and $\tau(x, y) < D_K$ there is a (timelike) geodesic joining them which is contained entirely in U.*

(iii) *Given two timelike geodesics $\alpha, \beta : [0,1] \to X$ of arbitrary time orientation emanating at $\alpha(0) = \beta(0) = x$, we have that $\tilde{\measuredangle}_x^{K,S}(\alpha(s), \alpha(t))$ is a monotonically increasing function in s and t (where it is defined).*

Note that by assuming that our space is regular, points *(i)* and *(ii)* above are precisely those given in Definition 2.7 of timelike curvature bounds and only point *(iii)* differs.

Currently, monotonicity comparison is the only formulation which is equivalent to triangle comparison in reasonable generality, with other formulations being implied by, but not implying, the monotonicity condition.[3]

Theorem 2.16 (Triangle and monotonicity comparison are equivalent)**.** *Let $K \in \mathbb{R}$ and let X be a regular Lorentzian pre-length space. Then X has timelike curvature bounded below by K in the sense of Definition 2.7 if and only if it satisfies K-monotonicity comparison from below.*

Proof. See [BS22, Theorem 4.13]. □

Theorem 2.17 (Curvature bounds imply angle and hinge comparison)**.** *Let X be a regular Lorentzian pre-length space with timelike curvature bounded below by $K \in \mathbb{R}$. Let $x \in X$ and let $\alpha, \beta : [0,1] \to X$ be any two timelike geodesics emanating from x.*

(i) *It holds that*

$$\measuredangle_x^S(\alpha, \beta) \leq \tilde{\measuredangle}_x^S(\alpha(s), \beta(t)) \tag{2.7}$$

for all s, t which form a timelike triangle with x.

(ii) *Let $(\bar{\alpha}, \bar{\beta})$ be a comparison hinge in $\mathbb{L}^2(K)$. Then*

$$\tau(\alpha(1), \beta(1)) \geq \bar{\tau}(\bar{\alpha}(1), \bar{\beta}(1)). \tag{2.8}$$

Proof. See [BS22, Corollaries 4.11 and 4.12]. □

We conclude this chapter with the following useful fact about angles.

Proposition 2.18. *Let X be a strongly causal and locally causally closed Lorentzian pre-length space with timelike curvature bounded below by $K \in \mathbb{R}$, and let $\alpha : [0,1] \to X$ be a timelike geodesic. Let $x = \alpha(t)$ for $t \in (0,1)$ and consider the restrictions $\alpha_- = \alpha|_{[0,t]}$ and $\alpha_+ = \alpha|_{[t,1]}$ as past-directed and future-directed geodesics emanating from x, respectively. Let β be a timelike geodesic emanating from x. Then $\measuredangle_x(\alpha_-, \beta) = \measuredangle_x(\alpha_+, \beta)$.*

[3] [BMS22, Theorem 14] shows that classical triangle comparison can be deduced from angle comparison in the case of lower timelike curvature bounds using additional assumptions on the behaviour of angles.

Proof. See [BS22, Corollary 4.7] (and [BS22, Lemma 4.10] for the existence of the angle). $\square$

2.2 Elementary concepts from metric geometry

We now turn to presenting some basic definitions from the realm of metric geometry. Most importantly, we would like to highlight the fundamental differences in the definitions of lengths and curvature bounds in the metric setting when compared to those Lorentzian pre-length spaces.

Definition 2.19 (Length of a curve and geodesics)**.** *Let (X,d) be a metric space. The* length *of a curve $\gamma:[a,b]\to X$ from x to y is defined as*

$$L_d(\gamma) := \sup\left\{ \sum_{i=0}^{n-1} d(\gamma(t_i),\gamma(t_{i+1})) \,\middle|\, a=t_0<t_1<\ldots<t_n=b, n\in\mathbb{N} \right\}. \tag{2.9}$$

If $L_d(\gamma)=d(x,y)$, then γ is called a distance-realizer *or* geodesic*. As we will generally not consider metric and Lorentzian geodesics simultaneously, context should be sufficient to deduce which of the two concepts is being applied and there should be no confusion between them.*

We define triangles, comparison triangles, and comparison points in complete analogy to Definition 2.5. Again we assume that all triangles satisfy size bounds. As discussed for geodesics, we will generally not use Lorentzian and Riemannian model spaces simultaneously, however, we shall denote the Riemannian model spaces by M_k, cf. [CE75].

Definition 2.20 (Metric triangle comparison)**.** *Let X be a metric space. An open subset U is called a $\geq k$-*comparison neighbourhood *(or $\leq k$-*comparison neighbourhood*) if $(U,d|_{U\times U})$ is geodesic for pairs of points with distance less than* diam(M_k)*, and for all triangles $\Delta(x,y,z)$ in U, and all $p,q\in\Delta(x,y,z)$, the following is satisfied: let $\bar\Delta(\bar x,\bar y,\bar z)$ be a comparison triangle in M_k for $\Delta(x,y,z)$ (satisfying size bounds) and let $\bar p,\bar q\in\bar\Delta(\bar x,\bar y,\bar z)$ be comparison points for p and q respectively. Then*

$$d(p,q)\geq d(\bar p,\bar q) \quad (\text{or } d(p,q)\leq d(\bar p,\bar q))\,. \tag{2.10}$$

We say X has curvature bounded below by *k if it is covered by $\geq k$-comparison neighbourhoods. Likewise, its* curvature is bounded above by *k if it is covered by $\leq k$-comparison neighbourhoods.*

We say X has global curvature bounded below (or above) by k if X is a $\geq k$ (or $\leq k$) comparison neighbourhood. Spaces with global curvature bounded above by k are called CAT(k) spaces.

3 Metric spaces with global curvature bounds

The globalization of curvature bounds in metric spaces serves as a natural motivation for investigating analogous results in the Lorentzian case. Hence, we include here a brief discussion of the metric picture in order to familiarize ourselves and the reader with the techniques and constraints we wish to transfer. For details of the wider metric setting, we refer the reader to [BBI01, BH99, AKP19].

3.1 Curvature bounded above

Let us first consider the case of curvature bounded above. The following example demonstrates that globalization is not automatic in this case, and that some additional assumptions are required.

Example 3.1 (The circle)**.** Consider the unit circle $X := \mathbb{S}^1$ with its intrinsic metric. Then locally, X is isometric to a line segment, which clearly has curvature bounded above by $k = 0$. However, X itself is not a ≤ 0-comparison neighbourhood: consider a large triangle defined by three equidistant points in X, such that it covers the whole circle. The corresponding comparison triangle in the Euclidean plane is also equilateral. Given any two points p, q on different sides of the triangle in X, the triangle inequality yields equality when going along the shorter of the two arcs between each pair of points, see Figure 3.1. However, in the plane, triangle equality is only obtained when the two points lie on the same side of the triangle. It follows that (2.10) is not satisfied. $\triangle$

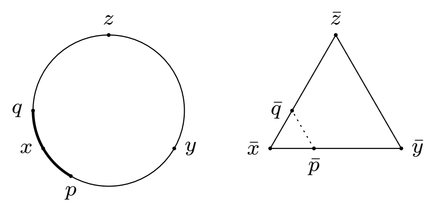

Figure 1: Triangle comparison fails for too large triangles in the circle.

The following theorem specifies sufficient additional assumptions under which upper curvature bounds may be globalized:

Theorem 3.2 (Alexandrov's Patchwork)**.** *Let X be a metric space with (local) curvature bounded above by k and suppose that there is a unique geodesic joining each pair of points that are a distance less than* $\mathrm{diam}(M_k)$ *apart. If these geodesics vary continuously with their endpoints,*[4] *then X is a CAT(k) space.*

[4]This means that if $x_n \to x, y_n \to y$, then $\gamma_{x_n y_n} \to \gamma_{xy}$ uniformly.

Proof. See [BH99, Proposition II.4.9]. $\square$

On a related note, it turns out that these additional assumptions under which the globalization of curvature bounds does work are rather mild, in the sense that they always hold in a $\mathrm{CAT}(k)$ space. In other words, a metric space with curvature bounded above by k is $\mathrm{CAT}(k)$ if and only if it is uniquely geodesic (for points with distance less than $\mathrm{diam}(M_k)$) and these geodesics vary continuously with their endpoints.

Proposition 3.3 (Elementary properties of $\mathrm{CAT}(k)$ spaces)**.** *Let X be a CAT(k) space. Then geodesics in X are unique (for points with distance less than $\mathrm{diam}(M_k)$) and these vary continuously with their endpoints.*

Proof. See [BH99, Proposition II.1.4]. $\square$

It should be immediately apparent that the circle with its intrinsic metric does not have continuously varying geodesics, let alone unique geodesics. Therefore, by the above proposition, it cannot be $\mathrm{CAT}(0)$.

3.2 Curvature bounded below

Concerning curvature bounded below, there is also a globalization result, which is perhaps even more iconic than the Alexandrov's Patchwork approach for curvature bounded above. It is known as the theorem of Toponogov and was first proven for general complete length spaces by Perelman in [BGP92].

Note that for Theorem 3.4 and Theorem 3.5 in the form stated, we follow the authors of [BBI01] and explicitly exclude 1-dimensional spaces. More precisely, if $k > 0$, then X must not be isometric to $\mathbb{R}$, $(0, \infty)$, $[0, B]$ for any $B > \frac{\pi}{\sqrt{k}}$, or any circle with radius greater than $\frac{2\pi}{\sqrt{k}}$.

Theorem 3.4 (Toponogov's Globalization Theorem)**.** *Let X be a complete length space with curvature bounded below by k, which is not one of the aforementioned 1-dimensional spaces. Then X has global curvature bounded below by k.*

Proof. See [BBI01, Theorem 10.3.1] for a proof under a local compactness assumption. See also [LS13, AKP19] for more general proofs, as well as a timeline of refinements. $\square$

So far, we have been unable to transport this result to the synthetic Lorentzian setting, but, as we shall discuss a little further in the outlook, we are actively working on formulating a result.

In the metric case, there is an addendum to Toponogov's Theorem, generalizing the Bonnet–Myers Theorem from Riemannian geometry to the setting of Alexandrov geometry. In essence, the theorem bounds the diameter of a complete length space with local, positive, lower curvature bound in such a way that comparison triangles exist, thus eliminating concern about the existence of triangles in X which are too large.

Theorem 3.5 (Lower curvature bounds imply finite diameter)**.** *Let X be a complete length space with (local) curvature bounded below by some $k > 0$, which is not one of the aforementioned 1-dimensional spaces. Then* $\mathrm{diam}(X) \leq \frac{\pi}{\sqrt{k}}$.

Proof. See [BBI01, Theorem 10.4.1]. □

Since the metric Bonnet–Myers Theorem is a direct result of Toponogov's Globalization Theorem 3.4, we will not be able to precisely follow the derivation given in [BBI01, Therem 10.3.1] when producing a Lorentzian analogue. However, we will derive a diameter bound on Lorentzian pre-length spaces, under the slightly stricter assumption of global curvature being bounded below by some $K < 0$, cf. Theorem 4.11. The corresponding result assuming only local curvature bounds is then a natural candidate for future research (see Section 5 for more detail).

4 Lorentzian pre-length spaces with global curvature bounds

We now return fully to the setting of synthetic Lorentzian geometry. As previously mentioned, the main task of this work is to provide Lorentzian versions of the Alexandrov's Patchwork approach to globalizing upper curvature bounds and the Bonnet–Myers Theorem constraining the diameter of spaces with positive lower curvature bounds on sectional curvature, cf. Theorem 3.2 and Theorem 3.5, respectively. Let us first discuss Alexandrov's Patchwork.

4.1 Timelike curvature bounded above

As it turns out, the proof of the metric globalization result, Theorem 3.2, may be very nicely adapted to the Lorentzian setting if we respect some minor technicalities. We first provide a few preparatory results, before diving into the proof proper. In particular, we will need the Gluing Lemma for timelike triangles:

Lemma 4.1 (Gluing Lemma for timelike triangles, Case I)**.** *Let X be a Lorentzian pre-length space and let $U \subseteq X$ be an open subset satisfying (i) and (ii) in the definition for a comparison neighbourhood in X, cf. Definition 2.7. Let $T_3 := \Delta(x, y, z)$ be a timelike triangle in U satisfying size bounds for $\mathbb{L}^2(K)$, with $K \in \mathbb{R}$ fixed but arbitrary. Let $p \in [x, z]$ be such that $p \ll y$ (or $y \ll p$). In other words, $T_1 := \Delta(x, p, y)$ and $T_2 := \Delta(p, y, z)$ are again*

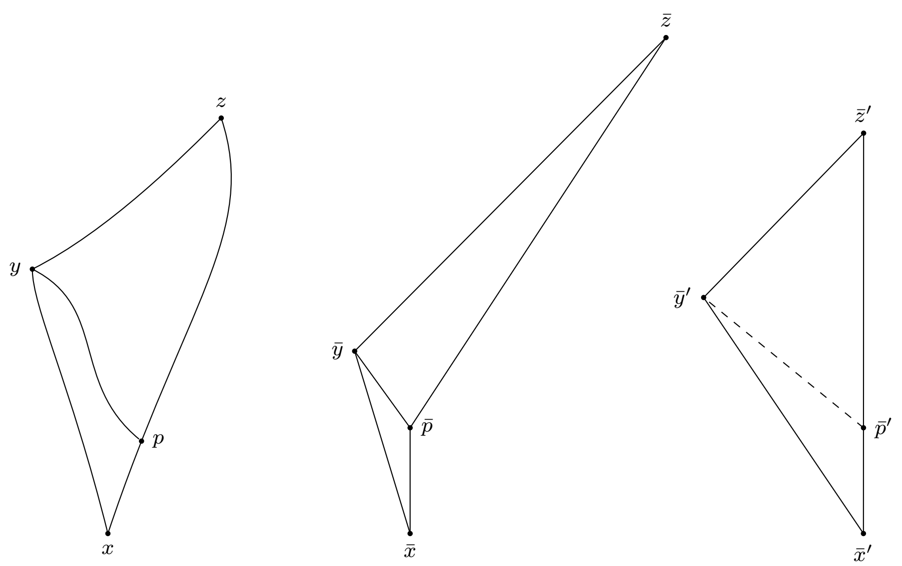

Figure 2: A timelike triangle in X subdivided into two timelike triangles, the comparison triangles for the smaller triangles and the comparison triangle for the big triangle.

timelike triangles (if $y \ll p$, it is necessary to interchange the order of y and p in the triangles), see Figure 2. Let $\overline{T}_1 := \Delta(\bar{x}, \bar{p}, \bar{y})$ and $\overline{T}_2 := \Delta(\bar{p}, \bar{y}, \bar{z})$ be comparison triangles for T_1 and T_2 in $\mathbb{L}^2(K)$, respectively. Suppose T_1 and T_2 satisfy timelike curvature bounds from above for K, i.e., for all $a, b \in T_i$ and corresponding comparison points $\bar{a}, \bar{b} \in \overline{T}_i, i = 1, 2$, we have $\tau(a, b) \geq \bar{\tau}(\bar{a}, \bar{b})$ (cf. Definition 2.7.(iii)). Then T_3 satisfies the same timelike curvature bound from above.

Proof. See [BR22, Lemma 4.3.1]. Note that this still works with the new definition of timelike curvature bounds. □

Lemma 4.2 (Gluing Lemma for timelike triangles, Case 2)**.** *Let X and U be as in Lemma 4.1 and let $\Delta(x, y, z)$ be a timelike triangle in U. Let p be a point on γ_{xy} (or γ_{yz}) and consider the two resulting sub-triangles that share the (timelike) segment γ_{pz} (or γ_{xp}). If the sub-triangles satisfy a timelike curvature bound from above, then so does the original triangle.*

Proof. See [BR22, Corollary 4.3.2]. □

So now we know that, if we can triangulate each big timelike triangle into smaller ones, with each of the small sub-triangles contained in a $\leq K$-comparison neighbourhood, then we can reconstruct the big timelike triangle step-by-step, using the Gluing Lemma to get that Definition 2.7.(iii) is satisfied by the big triangle.

In order to globalize curvature bounds, we now require conditions which guarantee such a triangulation of arbitrary big triangles, with sub-triangles contained in comparison neighbourhoods. Using Proposition 2.11, we have the following elegant description of comparison neighbourhoods in strongly causal Lorentzian pre-length spaces with curvature bound, cf. [Ber20, Remark 2.2.12], which shall turn out to be sufficient:

Proposition 4.3 (Timelike diamonds form neighbourhood basis of comparison neighbourhoods)**.** *Let X be a strongly causal and non-timelike locally isolating Lorentzian pre-length space with curvature bounded above (or below) by $K \in \mathbb{R}$. Then each point has a neighbourhood basis of timelike diamonds which are also comparison neighbourhoods.*

Proof. Let $x \in X$ and let U be a comparison neighbourhood of x. Any timelike diamond $D := I(p, q)$ containing x and contained in U (we can even assume $p, q \in U$) is a comparison neighbourhood: Indeed, points *(i)* and *(iii)* of Definition 2.7 are directly inherited when passing to open subsets, and *(ii)* follows by causal convexity of D, as in Lemma 2.12.

As X is strongly causal and non-timelike local isolating, Proposition 2.11 yields that timelike diamonds form a neighbourhood basis. Hence, for each neighbourhood V of x, there exists a timelike diamond containing x, which is contained in $U \cap V$. By the above, such a timelike diamond is also a comparison neighbourhood. $\square$

Definition 4.4 (Geodesic map)**.** *Let X be a uniquely geodesic and regular Lorentzian pre-length space. Viewing the timelike relation $\ll$ as a subset of $X \times X$, the* geodesic map[5] *of X is formally defined as*

$$G : \ll \times [0, 1] \to X\,, \quad G(x, y, t) := \gamma_{xy}(t)\,. \tag{4.1}$$

We say that geodesics vary continuously *if G is continuous.*

The definition above appears to be a different notion of continuous variation of geodesics to that required in Theorem 3.2. As it turns out, however, they are equivalent (note that of course in the Lorentzian formulation of the version used in Theorem 3.2 we must restrict to sequences and limits of timelike related points). The following proposition does not make use of τ at all, but we still formulate it in the Lorentzian context.

Proposition 4.5 (Equivalent notions of continuously varying geodesics)**.** *Let X be a uniquely geodesic and regular Lorentzian pre-length space. Then G is continuous if and only if (timelike) geodesics vary continuously in the sense of Theorem 3.2.*

[5]This is closely related to the *line-of-sight map*, cf. [AKP19, Definition 9.32].

Proof. First assume we have continuously varying geodesics in the sense of Theorem 3.2. Note that since geodesics are continuous by definition, we know that G is continuous in t. We have to show $G(x_n, y_n, t_n) \to G(x, y, t)$ for sequences $x_n \to x, y_n \to y, t_n \to t, x_n \ll y_n, x \ll y$, i.e., $\gamma_{x_n y_n}(t_n) \to \gamma_{xy}(t)$. We have $d(\gamma_{x_n y_n}(t_n), \gamma_{xy}(t)) \leq d(\gamma_{x_n y_n}(t_n), \gamma_{xy}(t_n)) + d(\gamma_{xy}(t_n), \gamma_{xy}(t))$ and conclude that the right hand goes to 0 using the uniform convergence $\gamma_{x_n y_n} \to \gamma_{xy}$ and the fact that geodesics are continuous.

Conversely, suppose G is continuous. Given sequences $x_n \to x, y_n \to y$, define $f_n : [0,1] \to \mathbb{R}, f_n(s) := d(\gamma_{x_n y_n}(s), \gamma_{xy}(s))$. The function f_n is continuous on compact domain, hence f_n attains its maximum at least once on $[0,1]$. Fix $s_n := \operatorname{argmax}(f_n(s))$ to be one such maximizing parameter (the precise choice is not important). We know $f_n(s) \to 0$ pointwise in s as $n \to \infty$ and we want to show that this convergence is uniform. As $\{s_n\}_n$ is contained in the compact set $[0,1]$, we may assume without loss of generality that $s_n \to s'$ for some $s' \in [0,1]$. Then

$$\begin{aligned} d(\gamma_{x_n y_n}(s), \gamma_{xy}(s)) = f_n(s) \leq f_n(s_n) &= d(\gamma_{x_n y_n}(s_n), \gamma_{xy}(s_n)) \\ &\leq d(\gamma_{x_n y_n}(s_n), \gamma_{xy}(s')) + d(\gamma_{xy}(s'), \gamma_{xy}(s_n)), \end{aligned}$$

which goes to 0 as $n \to \infty$ uniformly in s, using that both the geodesic map G and geodesics γ_{xy} are continuous in turn. □

We should modify the above proposition to continuous variation of the endpoints, up to length D_k.

Theorem 4.6 (Alexandrov's Patchwork Globalization, Lorentzian version)**.** *Let X be a strongly causal, non-timelike locally isolating, and regular Lorentzian pre-length space which has (local) timelike curvature bounded above by $K \in \mathbb{R}$. Suppose that X satisfies* (i) *and* (ii) *in Definition 2.7, i.e., $\tau^{-1}([0, D_K))$ is open and τ is continuous on that set, and for all $x \ll y$ in X with $\tau(x, y) < D_K$, there exists a geodesic joining them. Additionally assume that the geodesics between timelike related points with τ-distance less than D_K are unique. Let G be the geodesic map of X restricted to the set $\{(x, y, t) \in \ll \times [0,1] \,|\, \tau(x, y) < D_K\} = \tau^{-1}((0, D_K)) \times [0,1]$ and assume that G is continuous. Then X also satisfies Definition 2.7.*(iii)*, in particular it is a $\leq K$-comparison neighbourhood and X has global curvature bounded above by K.*

Proof. By our assumptions, it is only left to show triangle comparison. Let $\Delta(x, y, z)$ be a timelike triangle in X. Given $t \in [0,1]$, let $\beta_t := \gamma_{x \gamma_{yz}(t)} = G(x, \gamma_{yz}(t), \cdot) : [0,1] \to X$ be the geodesic from x to $\gamma_{yz}(t)$. By the continuity of G, we can regard the map $F(s, t) := \beta_t(s)$ as a geodesic variation with starting point x that "spans" the timelike triangle $\Delta(x, y, z)$. In particular, this "filled in" triangle is compact as the continuous image under F of the compact set $[0,1] \times [0,1]$. Fix $t \in [0,1]$. For each $s \in [0,1]$, we find a timelike diamond $I(x_s, y_s)$ that is a comparison neighbourhood of $\beta_t(s)$, cf. Proposition 2.11 and Proposition 4.3. Since β_t is continuous, there is a neighbourhood N_s of s in $[0,1]$ such that $\beta_t(N_s) \subseteq I(x_s, y_s)$. In particular, for $s \in (0,1)$, we find $s^- < s < s^+$ in N_s. By the causal convexity of diamonds, we then obtain that $I_s := I(\beta_t(s^-), \beta_t(s^+)) \subseteq I(x_s, y_s)$ is also a comparison neighbourhood of $\beta_t(s)$. The point is that we can choose the comparison neighbourhood diamonds in such a way that the governing points are situated on the geodesic. For the parameters 0 and 1 this will not be possible as these are the endpoints of the geodesic, however, we may still force one of the governing points to be on β_t, i.e., we set $I_0 := I(x_0, \beta_0(0^+))$ and $I_1 := I(\beta_1(1^-), y_1)$. Clearly, $\bigcup_s I_s$ is an open cover of $\beta_t([0,1])$. By compactness, we can extract a finite subcover[6]say $\bigcup_{k=0}^n I_{s_k} \supseteq \beta_t([0,1])$. Now order these diamonds with respect to, say, (the parameters of) their future governing points, i.e., $s_k^+ < s_{k+1}^+$ for all k. Further assume that the cover is minimal in the sense that no diamond can be removed from the cover; in particular, no diamond is entirely contained inside another one. This then immediately implies that the bottom governing points are ordered similarly and that subsequence diamonds overlap and only subsequent ones do so, i.e.,

$$I_i \cap I_j \neq \emptyset \iff |i-j| \leq 1. \tag{4.2}$$

Clearly, $F(\cdot, t) = \beta_t$. Since F is continuous and $\bigcup_{k=0}^n I_{s_k}$ is a neighbourhood of $\beta_t([0,1])$, it follows that there exists an open neighbourhood of t, denote it by J_t, such that $F([0,1], J_t) \subseteq \bigcup_{k=0}^n I_{s_k}$. By shrinking J_t if necessary, we can assume that $\gamma_{yz}(J_t) \subseteq I_1$. ♣ `Is this not automatically true by the next step?` ♣ Visually, $\bigcup_{k=0}^n I_{s_k}$ covers $\beta_{t'}$ for all t' in a neighbourhood of t and each $\beta_{t'}$ ends in the top diamond I_1.

Now we let t vary: doing the above described procedure for each $t \in [0,1]$, we end up with an open cover $\bigcup_t J_t$[7] of $[0,1]$. Again by a compactness argument, we can extract a finite subcover from $\bigcup_t J_t$, say $\bigcup_{l=0}^m J_{t_l} \supseteq [0,1]$. Order these set in the same way as the diamonds, i.e., increasing with respect to, say, the right endpoint, and remove unnecessary ones. Then also the left endpoints are ordered in an increasing fashion and subsequent sets do overlap and these are the only ones which overlap. In total, we obtain, modifying the above notation slightly, that

$$\bigcup_{l=0}^{m} \bigcup_{k=0}^{n_l} I_{s_k}^{t_l} \supseteq F([0,1],[0,1]), \tag{4.3}$$

[6]Note that because of the way we chose the governing points of each I_s, the first and the last diamonds I_0 and I_1 are always included (because the endpoints are not inside any other I_s).

[7]Note that similar to I_0 and I_1 from above, the sets J_0 and J_1 contain 0 and 1 at the boundary, respectively. Visually, this means that at the edges of the original triangle, nearby geodesics can only be "on one side" of these edges.

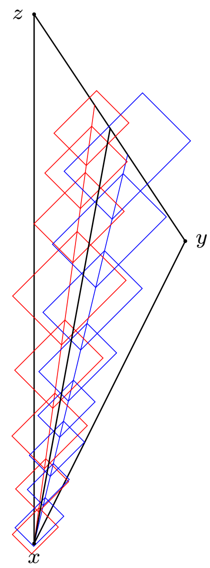

Figure 3: J_{t_l} (t_l represented by the blue geodesic β_{t_l}) and $J_{t_{l+1}}$ (t_{l+1} represented by the red geodesic $\beta_{t_{l+1}}$) overlap: $\tilde{t}_l \in J_{t_l} \cap J_{t_{l+1}}$ ($\tilde{t}_l$ represented by the black geodesic $\beta_{\tilde{t}_l}$).

where I_s^t is the diamond "around" $\beta_t(s)$, i.e., the t emphasizes that this diamond belongs to the cover of β_t. The reward for this tedious construction is now the following: The triangle may be viewed as a fan consisting of m pieces, and the covers of subsequent "fan-geodesics" share some geodesics between them. More precisely, as $J_{t_l} \cap J_{t_{l+1}} \neq \emptyset, l = 0, \dots, m-1$, and all geodesics ending in $\gamma_{yz}(J_{t_l})$ are contained in $\bigcup_{k=0}^{n_l} I_{s_k}^{t_l}$ (and the same for $l+1$), we know there exists some $\tilde{t}_l$ such that

$$\beta_{\tilde{t}_l}([0,1]) \subseteq (\bigcup_{k=0}^{n_l} I_{s_k}^{t_l}) \cap (\bigcup_{k=0}^{n_{l+1}} I_{s_k}^{t_{l+1}}). \tag{4.4}$$

A sketch of this process is depicted in Figure 3. We continue with the process of triangulation as follows: given l, consider the timelike triangle $\Delta(x, \beta_{\tilde{t}_l}(1), \beta_{t_{l+1}}(1))$. By construction, both $\beta_{\tilde{t}_l}$ and $\beta_{t_{l+1}}$ end in $\gamma_{yz}(J_{t_{l+1}}) \subseteq I_{s_{n_{l+1}}}^{t_{l+1}}$ and enter that set via $I_{s_{n_{l+1}}}^{t_{l+1}} \cap I_{s_{n_{l+1}-1}}^{t_{l+1}}$, i.e., they pass through the intersection of the ultimate and penultimate diamonds covering $\beta_{t_{l+1}}$. In particular, we can choose $\tilde{r}_1$ such that $\beta_{\tilde{t}_l}(\tilde{r}_1)$ is in said intersection. Note that the top governing point of the second to last diamond is timelike after the chosen point on $\beta_{\tilde{t}_l}$, i.e.,

$$\beta_{\tilde{t}_l}(\tilde{r}_1) \ll \beta_{t_{l+1}}(s^+_{n_{l+1}-1}). \tag{4.5}$$

By the openness of $\ll$, we can move a bit below the top governing

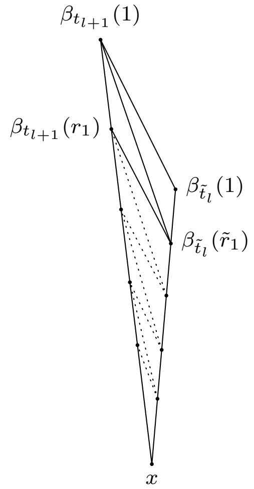

Figure 4: The process of subdividing a slim triangle.

point and still retain a timelike relation to the chosen point on $\beta_{\tilde{t}_l}$. In particular, both of these points are then contained in the intersection of the last two diamonds, and by the causal convexity also their connecting geodesic is entirely contained therein. More precisely, we find r_1 such that $\beta_{t_{l+1}}(r_1) \in I^{t_{l+1}}_{s_{n_{l+1}}} \cap I^{t_{l+1}}_{s_{n_{l+1}-1}}$ and $\beta_{\tilde{t}_l}(\tilde{r}_1) \ll \beta_{t_{l+1}}(r_1)$. Essentially, we constructed a quadrilateral consisting of $\beta_{\tilde{t}_l}(\tilde{r}_1), \beta_{t_{l+1}}(r_1), \beta_{\tilde{t}_l}(1)$ and $\beta_{t_{l+1}}(1)$, which is completely contained in $I^{t_{l+1}}_{s_{n_{l+1}}}$. By transitivity of $\ll$, also the "past most" and "future most" points of this quadrilateral are timelike related, so we can split this into two timelike triangles $\Delta(\beta_{\tilde{t}_l}(\tilde{r}_1), \beta_{t_{l+1}}(r_1), \beta_{t_{l+1}}(1))$ and $\Delta(\beta_{\tilde{t}_l}(\tilde{r}_1), \beta_{\tilde{t}_l}(1), \beta_{t_{l+1}}(1))$, see Figure 4. Both of these triangles are entirely contained in the last timelike diamond $I^{t_{l+1}}_{s_{n_{l+1}}}$, so they satisfy the curvature bound by assumption. Iteratively continuing this procedure, we end up at x after a finite amount of steps. As $\beta_{\tilde{t}_l}(\tilde{r}_1) \ll \beta_{t_{l+1}}(r_1)$ are also contained in $I^{t_{l+1}}_{s_{n_{l+1}-1}}$, we can continue this procedure iteratively. After $n_{l+1}-1$ steps, we end up with $\beta_{\tilde{t}_l}(\tilde{r}_{n_{l+1}-1}) \ll \beta_{t_{l+1}}(r_{n_{l+1}-1})$ lying in $I^{t_{l+1}}_{s_1}$, where also x lies.

That is, in the end we have $n_{l+1} - 1$ quadrilaterals, each of which we can split into two timelike triangles, and one additional timelike triangle at the bottom ending in x. Each of these timelike triangles is contained in one of the comparison neighbourhoods $\{I^{t_{l+1}}_{s_i} : i = 1, \cdots, n_{l+1}\}$ (i.e. the chain of comparison diamonds covering the geodesic $\beta_{t_{l+1}}$ and $\beta_{\tilde{t}_l}$), so satisfies the curvature bound, hence several applications of the Gluing Lemma 4.1 yield that the "long and slim" triangle $\Delta(x, \beta_{\tilde{t}_l}(1), \beta_{t_{l+1}}(1))$ satisfies the curvature bound. Note that in a triangle of the form $\Delta(x, \beta_{t_l}(1), \beta_{\tilde{t}_{l+1}}(1))$ the top side has a different time orientation from the point of view of the geodesic β_{t_l} around which the covering is centered, but this changes nothing for the above described process. So we can do this for all of the $2m-1$ long and slim triangles and apply the Gluing Lemma $2m-2$ times to obtain that the original triangle $\Delta(x,y,z)$ obeys the desired curvature bound. $\square$

At first glance, our proof appears to be quite similar to the metric version in [BH99, Proposition II.4.9]. There are however, some technical details we have to be wary of. In particular, in an arbitrary covering of the triangle, even if we use timelike diamonds, it is generally not true that we can achieve a subdivision consisting of timelike triangles such that each sub-triangle is contained within a comparison neighbourhood. We have to carefully construct the covering and then construct the sub-triangles in a seemingly complicated way as well.

As with the circle in the metric case (see Example 3.1), it is possible to construct counterexamples to the automatic globalization of upper curvature bounds on Lorentzian pre-length spaces. The following is one such example, where the space has (local) curvature bounded above, but has neither unique nor continuously varying geodesics:

Example 4.7 (The Lorentzian cylinder)**.** We set the Lorentzian cylinder to be the spacetime $X=\mathbb{R}\times\mathbb{S}^1$, i.e., take a strip $\mathbb{R}\times[0,2\pi]$ in Minkowski space and glue the boundary as depicted by the arrows in Figure 5(which is not to be confused with the totally vicious cylinder $\mathbb{S}^1\times\mathbb{R}$!).

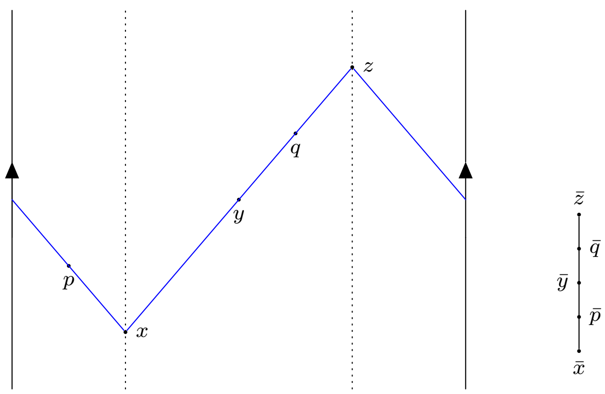

Figure 5: The Lorentzian cylinder. The depicted triangle fails to satisfy an upper curvature bound since its comparison triangle is degenerate.

This space is locally isometric to Minkowski space, hence clearly has (local) timelike curvature bounded above by 0.

Take two (dotted) vertical lines on the cylinder, which are directly opposite each other (as in Figure 5). Given two timelike related points, with one point on each of the two vertical lines, there exist precisely two geodesics between these points (wrapping around to the right and to the left on the cylinder, respectively). Consider any such pair of points, denoted x and

z, and the corresponding pair of geodesics, and choose as a third vertex some point y on one of the two geodesics. Then the comparison triangle for $\Delta(x, y, z)$ is clearly degenerate. Choosing two points p and q on the two different geodesics (and at different parameters, say p occurs at an earlier parameter than q) sufficiently far away from the endpoints of the two geodesics, p and q will not be timelike related, i.e., $\tau(p, q) = 0$. However, as the comparison triangle is essentially a line segment and the two points are at different parameters, we clearly have $\tau(\bar{p}, \bar{q}) > 0$, violating global upper curvature bounds. △

However, as in the metric case, some of our additional assumptions are relatively mild and can be recovered from the curvature bounds. More specifically, while we have the following result concerning the uniqueness of geodesics, we do not yet know under which conditions the geodesic map G is continuous.

Proposition 4.8 (Unique geodesics in upper curvature bounds)**.** *Let X be a strongly causal Lorentzian pre-length space with timelike curvature bounded above by $K \in \mathbb{R}$. Let $x \ll y$ be in a comparison neighbourhood $U \subseteq X$ and suppose $\tau(x, y) < D_K$. Then there exists a unique geodesic from x to y contained in U. In particular, if X satisfies a global upper curvature bound, geodesics between timelike related points in X with τ-distance less than D_K are unique.*

Proof. Suppose towards a contradiction that there is another geodesic from x to y. Denote the curves corresponding to two of these geodesic segments by α_1 and α_2, respectively, and parameterize them with constant speed on $[0, 1]$. Let $p \in \alpha_1((0, 1))$ and consider the timelike triangle $\Delta(x, p, y)$, which satisfies size-bounds for $\mathbb{L}^2(K)$. Clearly the comparison triangle in $\mathbb{L}^2(K)$ is degenerate. As $\alpha_1 \neq \alpha_2$, there exists $t \in (0, 1)$ such that $\alpha_1(t) \neq \alpha_2(t)$. Then there exist neighbourhoods V_1 and V_2 of $\alpha_1(t)$ and $\alpha_2(t)$, respectively, such that $V_1 \cap V_2 = \emptyset$. As X is strongly causal, we find points $x_1, y_1, \ldots, x_n, y_n$ and $p_1, q_1, \ldots, p_m, q_m$ such that $\alpha_1(t) \in U_1 := \bigcap_{i=1}^n I_1(x_i^1, y_i^1) \subseteq V_1$ and $\alpha_2(t) \in U_2 := \bigcap_{j=1}^m I_1(p_j^2, q_j^2) \subseteq V_2$. By the continuity of α_1, there is some neighbourhood I_1 of t such that $\alpha_1(I_1) \subseteq U_1$. As U_1 is causally convex by definition, all diamonds with endpoints inside U_1 are contained in U_1. Similarly for U_2. In particular, there exists $\varepsilon > 0$ such that $D_1 := I(\alpha_1(t - \varepsilon), \alpha_1(t + \varepsilon)) \subseteq U_1$ and $D_2 := I(\alpha_2(t - \varepsilon), \alpha_2(t + \varepsilon)) \subseteq U_2$. Then $\alpha_2(t) \notin D_1$ (and vice versa), so either $\alpha_1(t - \varepsilon) \not\ll \alpha_2(t)$ or $\alpha_2(t) \not\ll \alpha_1(t + \varepsilon)$. However, in $\mathbb{L}^2(K)$ we have $\overline{\alpha}_1(s) = \overline{\alpha}_2(s)$ for all $s \in [0, 1]$. In particular, $\overline{\alpha}_1(t - \varepsilon) \ll \overline{\alpha}_2(t) \ll \overline{\alpha}_1(t + \varepsilon)$, a contradiction to upper timelike curvature bounds. □

Remark 4.9 (Globalization of continuity)**.** If we strengthen the requirements on X and the geodesic map, instead imposing that geodesics between any $x \ll y$ exist and are unique, and that the geodesic map G on its full domain $\ll \times [0,1]$ is continuous, it follows that τ is globally continuous (and indeed finite by [KS18, Lemma 2.25]). In particular, if τ is globally continuous, then X automatically satisfies condition *(i)* of Definition 2.7 and we need not assume so in Theorem 4.6.

As X has timelike curvature bounded above in the sense of Definition 2.7, it follows that τ is locally continuous[8] (with respect to the covering of comparison neighbourhoods constructed in Lemma 2.12). By using that X is geodesic and slightly adapting the proof of [KS18, Proposition 3.17], it can be shown that L_τ is upper semi-continuous. Combining this with the continuity of G, we find $L_\tau(G(x,y,\cdot)) = L_\tau(\gamma_{xy}) = \tau(x,y)$, hence τ is both upper and lower semi-continuous on X and is therefore globally continuous. $\triangle$

4.2 Timelike curvature bounded below

Finally, we show that a bound may be placed on the (finite) diameter (see Definition 2.6) of a Lorentzian pre-length space with negative lower timelike curvature bound. This result is in the spirit of the Bonnet–Myers Theorem (Theorem 3.5) from Riemannian geometry, however, as for previous results, we need to be careful about the Lorentzian subtleties.

First, note that due to the way timelike curvature bounds were introduced in [KS18], the hierarchy of curvature bound implications is reversed. More precisely, recall that if a metric space has curvature bounded above by some k, then it also has curvature bounded above by all $k' \geq k$. Similarly, if it has k as a lower curvature bound, it also has any $k' \leq k$ as a lower curvature bound. In the Lorentzian case, it turns out that any Lorentzian pre-length space satisfying timelike curvature bounded below by K, does so for all $K' \geq K$. Hence, we shall be required to assume a negative *lower* bound. In addition, due to the behaviour of Anti-deSitter discussed before Definition 2.6, we consider the finite diameter of Lorentzian pre-length spaces, rather than the ordinary diameter as in the metric case. Before diving into the theorem, we mention a lemma, giving a non-degeneracy condition for sub-triangles.

Lemma 4.10 (Non-degeneracy condition)**.** *Let X be a strongly causal, locally causally closed, regular, and geodesic Lorentzian pre-length space X and let U be a comparison neighbourhood in X. Let $a \ll b$ in U and let α be a geodesic in U starting at a and ending at b. Let $x = \alpha(t)$ and let $y \in I(x,b)$. Assume that both $\Delta(a,x,y)$ and $\Delta(x,y,b)$ satisfy size bounds. Let β be a timelike geodesic starting in x and ending in y, and denote by*

[8] In essence, we pass from local curvature bounds in the sense of Definition 2.7 to those in the sense of [KS18, Definition 4.7], where globalization of continuity is straightforward under our new assumptions.

$\alpha_- = \alpha|_{[0,t]}$ *and* $\alpha_+ = \alpha|_{[t,1]}$ *the parts of* α *in the past and future of* y*, respectively.*

(i) If X *has timelike curvature bounded below by* K *and* $\Delta(x, y, b)$ *is non-degenerate, then* $\Delta(a, x, y)$ *is also non-degenerate, and* $\measuredangle_x(\alpha_+, \beta)$ *and* $\measuredangle_x(\alpha_-, \beta)$ *are equal and positive.*

(ii) If X *has timelike curvature bounded above by* K *and the angle* $\measuredangle_x(\alpha_-, \beta)$ *exists and* $\Delta(a, x, y)$ *is non-degenerate, then* $\Delta(x, y, b)$ *and (if it satisfies size bounds)* $\Delta(a, y, b)$ *are also non-degenerate, and both* $\measuredangle_x(\alpha_\pm, \beta) > 0$ *though they need not be equal.*

Proof. (i) As $\Delta(x, y, b)$ is non-degenerate, also the corresponding comparison triangle is non-degenerate and hence $\tilde{\measuredangle}_x(y, b) > 0$. By angle comparison, cf. Theorem 2.17.*(i)*, we get $0 < \tilde{\measuredangle}_x(y, b) \leq \measuredangle_x(\alpha_+, \beta)$. As X is locally causally closed, strongly causal and has timelike curvature bounded below, we can apply Proposition 2.18, from which it follows that $\measuredangle_x(\alpha_+, \beta) = \measuredangle_x(\alpha_-, \beta) > 0$, and again by angle comparison, we have $\tilde{\measuredangle}_x(a, y) \geq \measuredangle_x(\alpha_-, \beta) > 0$. In particular, also $\Delta(a, x, y)$ is non-degenerate.

(ii) For the curvature bounded above case, the arguments of the curvature bounded below case reverse (we only get $\tilde{\measuredangle}_x(a, y) \leq \measuredangle_x(\alpha_-, \beta)$ from the triangle inequality of angles, see [BS22, Theorem 4.5.(i)]), and monotonicity comparison at the angle at a yields the statement on the big triangle.

□

The result of Theorem 4.11 should be closely compared to [CM20, Proposition 5.10]. In this pioneering work, the authors introduce synthetic Ricci curvature bounds using optimal transport methods, hence their result might be even closer in spirit to the original Bonnet–Myers Theorem than the one shown below. Moreover, should it prove true that Ricci curvature bounds (using optimal transport) are weaker than sectional curvature bounds (using triangle comparison) in the Lorentzian picture, as is the case for metric curvature comparison, cf. [Pet19], then our result has narrower scope. However, as the hierarchy of curvature bounds is not yet known and our method is distinct, the proof of the following theorem is valuable in its own right.

Theorem 4.11 (Bound on the finite diameter)**.** *Let* X *be a strongly causal, locally causally closed, regular, and geodesic*[9] *Lorentzian pre-length space which has global curvature bounded below by* K*. Assume* $K < 0$*. Assume that* X *possesses the following non-degeneracy condition: for each pair of*

[9]Global curvature bounds guarantee the existence of a geodesic for all $a \ll b$ with $\tau(a, b) < D_K$. In this context, however, we will need the existence of geodesics for all timelike related pairs of points slightly further apart. X being geodesic is a sufficient condition for this.

points $x \ll z$ in X we find $y \in X$ such that $\Delta(x, y, z)$ is a non-degenerate timelike triangle.[10] *Then* $\operatorname{diam}_{\text{fin}}(X) \leq D_K$.

Proof. Without loss of generality, we only consider $K = -1$. Let indirectly $a, b \in X$ with $\tau(a, b) = \pi + \varepsilon$ for some small enough $\varepsilon > 0$, and let $\alpha : [0, \pi + \varepsilon] \to X$ be a timelike distance realizer from a to b parameterized by τ-arclength. Let $x = \alpha(t_-)$ and $z = \alpha(t_+)$ for $t_- = \frac{\pi}{2} + \frac{\varepsilon}{2}$ (i.e., the midpoint), $t_+ = \frac{\pi}{2} + \frac{\pi}{8}$. Note that the specific value of t_+ is not important, any $t_+ \in (t_-, \pi)$ suffices. The corresponding point z is mainly used further on in the proof to ensure that a triangle with longest side shorter than $\tau(a, z) < \pi$, is realizable in $\mathbb{L}^2(-1)$. Denote by $\alpha_- = \alpha|_{[0,t_-]}$ and $\alpha_+ = \alpha|_{[t_-,\pi+\varepsilon]}$ the parts of α in the past and future of y, respectively.

By the non-degeneracy assumption on X, we find a point $y \in I(x, z)$ such that $\tau(x, z) > \tau(x, y) + \tau(y, z)$, i.e., $\Delta(x, y, z)$ is non-degenerate. Let β be a distance realizer from x to y. By Lemma 4.10.*(i)*, we get that both $\Delta(a, x, y)$ and $\Delta(x, y, z)$ are non-degenerate, and $\omega := \measuredangle_x(\alpha_-, \beta) = \measuredangle_x(\alpha_+, \beta) > 0$. We now claim that $\tau(a, b) < \tau(a, y) + \tau(y, b)$, contradicting the reverse triangle inequality.

We name the lengths: $t_- = \tau(a, x) = \tau(x, b) =: t$, $\tau(a, y) =: p$, $\tau(y, b) =: q$ and $\tau(x, y) =: m$, so the claim reads

$$2t < p + q. \tag{4.6}$$

We create a situation in $\mathbb{L}^2(K)$ consisting of comparison hinges for (α_-, β) and (β, α_+), giving the triangles $\Delta(\tilde{a}, \tilde{x}, \tilde{y})$ and $\Delta(\tilde{x}, \tilde{y}, \tilde{b})$, see Figure 6. Note that these triangles are non-degenerate since $\omega > 0$. We name the side-lengths $\tau(\tilde{a}, \tilde{y}) = \tilde{p}$, $\tau(\tilde{y}, \tilde{b}) = \tilde{q}$.

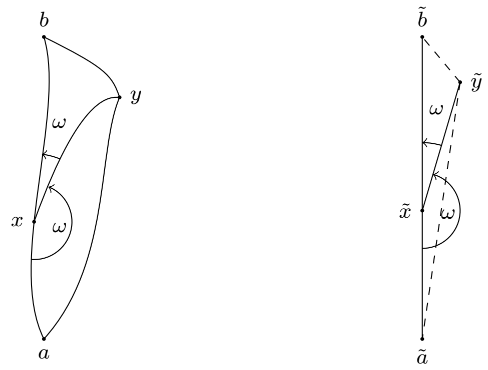

Figure 6: The construction in X and comparison hinges.

[10]This condition ensures the space is not locally 1-dimensional; Compare this to the discussion before Theorem 3.4 in the metric setting.

By hinge comparison, cf. Theorem 2.17.(ii), we get $p = \tau(a,x) \geq \tau(\tilde{a},\tilde{x}) = \tilde{p}$ and $q = \tau(x,b) \geq \tau(\tilde{x},\tilde{b}) = \tilde{q}$. We claim that $2t < \tilde{p} + \tilde{q}$. As $p \geq \tilde{p}$ and $q \geq \tilde{q}$, this implies the above claim.

By the reverse triangle inequality, we have $t > m + \tilde{q}$ and $\tilde{p} > t + m > 2m + \tilde{q}$, thus $\tilde{p} - \tilde{q} > 2m$. Note that the reverse triangle inequality yields strict inequalities since the triangles are non-degenerate. Recall that $\tilde{p} \leq p = \tau(a,y) \leq \tau(a,z) = \frac{\pi}{2} + \frac{\pi}{8}$ and $\tilde{q} \geq q = \tau(y,b) \geq \tau(z,b) = \frac{\pi}{2} + \varepsilon - \frac{\pi}{8} > \frac{\pi}{2} - \frac{\pi}{8}$. Thus, we get $0 < 2m < \tilde{p} - \tilde{q} < \frac{\pi}{4}$. In particular, as cosine is decreasing, we have

$$0 < \cos(\frac{\tilde{p} - \tilde{q}}{2}) < \cos(m). \tag{4.7}$$

We now write down the equations for $\omega = \measuredangle_{\tilde{x}}(\tilde{a},\tilde{y}) = \measuredangle_{\tilde{x}}(\tilde{y},\tilde{b})$ in the law of cosines (cf. [BS22, Lemma 2.4] and remember $K = -1$):

$$\begin{aligned} \cos(m)\cos(t) - \sin(m)\sin(t)\cosh(\omega) &= \cos(\tilde{p}), \\ \cos(m)\cos(t) + \sin(m)\sin(t)\cosh(\omega) &= \cos(\tilde{q}). \end{aligned}$$

We add these two equations to eliminate ω and then use the cosine addition formula:

$$\cos(m)\cos(t) = \frac{1}{2}(\cos(\tilde{p}) + \cos(\tilde{q})) = \cos(\frac{\tilde{p} + \tilde{q}}{2})\cos(\frac{\tilde{p} - \tilde{q}}{2}). \tag{4.8}$$

We know further reformulate and end up with

$$\cos(t)\frac{\cos(m)}{\cos(\frac{\tilde{p}-\tilde{q}}{2})} = \cos(\frac{\tilde{p} + \tilde{q}}{2}). \tag{4.9}$$

As $0 < \cos(\frac{\tilde{p}-\tilde{q}}{2}) < \cos(m)$, we know that the fraction on the left hand side is bigger than one, and since $\frac{\pi}{2} < t = \tau(a,x) = \tau(x,b) < \pi$, we have $\cos(t) < 0$. In total, we get

$$\cos(\frac{\tilde{p} + \tilde{q}}{2}) < \cos(t), \tag{4.10}$$

and as cos is monotonically decreasing, we obtain

$$\tilde{p} + \tilde{q} > 2t, \tag{4.11}$$

which by above arguments implies the original claim (4.6). □

Remark 4.12 (Global hyperbolicity and spacetimes)**.** If we additionally assume that X is a globally hyperbolic Lorentzian length space, then the result can be extended to apply to the diameter, in addition to the finite diameter. Indeed, τ is then finite, cf. [KS18, Theorem 3.28], so the diameter and the finite diameter agree. Our result may then be viewed as an extension of [BE79, Theorem 9.5], in which a bound is derived for the diameter of a globally hyperbolic spacetime with timelike sectional curvature bounded below by $K < 0$. △

Remark 4.13 (An implication of Theorem 4.11)**.** There is an immediate corollary to the metric version Theorem 3.5, stating that the perimeter of any triangle in a space with curvature bounded below by k cannot be greater than $\frac{2\pi}{\sqrt{k}}$, see [BBI01, Corollary 10.4.2]. This can be argued using hinge comparison. Typically, the corresponding Lorentzian version is at least as difficult as the metric result, so it is noteworthy that this corollary is easier in the Lorentzian world. In fact, the result immediately follows from the reverse triangle inequality: let $\Delta(x, y, z)$ be a timelike triangle, then by Theorem 4.11, we know $D_K > \tau(x, z) \geq \tau(x, y) + \tau(y, z)$, hence $\tau(x, y) + \tau(y, z) + \tau(x, z) < 2D_K$ as required. △

5 Outlook

Finally, let us discuss potential future research stemming from this paper. In particular, we wish to highlight the possibility of a Lorentzian version of the famous Toponogov Globalization Theorem (Theorem 3.4) for lower curvature bounds. In the smooth Lorentzian case, this was achieved in [Har82], however, despite attempts by the authors and several other researchers, a synthetic Lorentzian equivalent has not yet been obtained. Given the recent work of [BMS22, BS22] formulating curvature bounds on Lorentzian pre-length spaces in terms of monotonicity and angles, it is worth re-examining how well the techniques utilized in the proof of [Har82] adapt to our setting. As [BS22] also introduces curvature comparison of Lorentzian pre-length spaces via hinges, the proof of the metric statement provided in [AKP19, Theorem 8.31] may also be a good point of ingress. Most promisingly, the first and third author of this work are currently part of a collaboration developing a Lorentzian description of (lower) curvature bounds using the so-called four point condition, see [BBI01, Proposition 10.1.1]. This condition is used in Perelman's proof of the Toponogov Theorem for complete length spaces, cf. [BGP92, Section 3.4] (see also [BBI01, Theorem 10.3.1] for more clarification) so is likely to be a strong tool in the arsenal of synthetic Lorentzian geometry.

The authors of the current work have also been investigating generalizations of Theorem 4.11 which require only local curvature bounds, in the spirit of the metric Bonnet–Myers Theorem 3.5. Direct approaches have not yet proven fruitful. However, in the metric case, such a bound on the diameter of a (complete) length space is a direct consequence of Toponogov's Theorem 3.4, hence the space having local curvature bounded below is sufficient [BBI01, Theorem 10.4.1]. It is therefore plausible that the the statement of Theorem 4.11 could be strengthened in a similar manner, once a Toponogov style result is obtained in the Lorentzian pre-length space setting.

Another area of interest is the study of Lorentzian polyhedral spaces.

In metric geometry, it is known that one-dimensional polyhedral spaces, or (metric) graphs, which are additionally locally finite and connected, satisfy a local non-positive upper curvature bound. Furthermore, compact length spaces can be described by the Gromov–Hausdorff limit of finite graphs [BBI01, Proposition 7.5.5], performing a discretization of a continuous space. In the theory of quantum gravity, locally finite sets, now equipped with a partial order, are considered similarly when modelling discrete spacetimes [BLMS87, Sur19, DHS04]. Such 'causal sets' are represented by locally finite, transitively reduced, directed, acyclic graphs called Hasse diagrams, where the additional qualifiers reflect the properties of the inferred Lorentzian geometry. It is known that not all causal sets have continuum approximation which is given by a spacetime (see the excellent review paper [Sur19] for more details), however, it is plausible that a wider range of causal sets are approximated by a Lorentzian pre-length space. Conversely, under some additional constraints, it is expected that Lorentzian pre-length spaces are given by the Gromov–Hausdorff limit of causal sets, interpreted as polyhedral spaces, and that causal sets satisfy a timelike curvature bound. This would drive the development of the Lorentzian pre-length space and causal set frameworks forward in parallel, enabling consolidation and verification of derived results.

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