The Geometry of Monge–Ampère Equations in Fluid Dynamics

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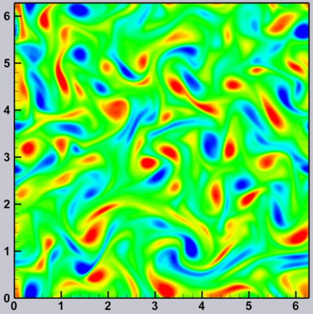
Motivation: Turbulence and Vortices

- Turbulent flows consist of complex interactions of vortex structures.
- In 2D, they combine as they evolve, forming stable coherent structures characterised by circulation/elliptic motion.
- In 3D, one finds knotted/linked tubes which accumulate at small scale. “sinews of turbulence.” [Moffatt et al. 1994]



Vorticity of evolving 2d turbulence at early time (Andrey Ovsyannikov - Ecole Centrale de Lyon)

Outline of Talk

1. Monge–Ampère Equations in Fluid Dynamics
2. Differential Geometry: An Overview
3. Monge–Ampère Geometry and 2D Navier–Stokes Flows
4. The Lychagin–Rubtsov Metric and its Pullback
5. Conclusion and Outlook

1. Monge–Ampère Equations in Fluid Dynamics

(Contact) Monge–Ampère Equations

- Are non-linear second-order PDEs which are quasi-linear w.r.t second order partial derivatives, up to determinants of the Hessian or its minors.
- In two dimensions, they take the form

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0\,.$$

where $A, B, \ldots E$ can depend on $x, y, \psi, \psi_x, \psi_y$ non-linearly.

- If $A, B, \ldots E$ do not depend on ψ, we have a symplectic Monge–Ampère (MA) equation.

Examples of Monge–Ampère Equations

Key linear examples:

- Laplace: $\Delta\psi = 0$
- Wave: $\Box\psi = 0$

From (3D) semi-geostrophic theory:

- Ertel: $\det(\mathrm{Hess}(P)) = q_g$
- Chynoweth–Sewell: $q_g(T_{xx}T_{yy} - (T_{xy})^2) + T_{zz} = 0$

Here, q_g is potential vorticity, P is a (modified) geopotential, and T is its partial Legendre dual with respect to x and y.

[Chynoweth and Sewell 1989, D'Onofrio et al 2023]

From Navier–Stokes to Poisson

- Homogeneous, incompressible Navier–Stokes equations on $\mathbb{R}^m$

$$\frac{\partial v}{\partial t} = -(v \cdot \nabla)v - \nabla p + \nu \Delta v \, (-c) \, .$$

- Taking the divergence and applying $\nabla \cdot v = 0$ one finds

$$\Delta p = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij}$$

where $\zeta_{ij} = \frac{1}{2}(\nabla_j v_i - \nabla_i v_j)$ and $S_{ij} = \frac{1}{2}(\nabla_j v_i + \nabla_i v_j)$.

- Vorticity term dominates $\Leftrightarrow$ $\Delta p > 0$.
Strain term dominates $\Leftrightarrow$ $\Delta p < 0$.

Poisson Equation in Two Dimensions

- In 2D, one has a stream function $v_1 = -\psi_y$ and $v_2 = \psi_x$.
- Poisson equation is a MA equation for the stream function

$$\frac{\Delta p}{2} = \left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) .$$

- *Vorticity dominates* $\Leftrightarrow$ $\Delta p > 0$ $\Leftrightarrow$ *Elliptic equation.*
  Strain dominates $\Leftrightarrow$ $\Delta p < 0$ $\Leftrightarrow$ *Hyperbolic equation.*
  No dominance $\Leftrightarrow$ $\Delta p = 0$ $\Leftrightarrow$ *Parabolic equation.*
  [Weiss 1991, Larchevêque 1993]

Poisson Equation for Pressure

"(This) equation for the pressure is by no means fully understood and locally holds the key to the formation of vortex structures through the sign of the Laplacian of the pressure. In this relation... may lie a deeper knowledge of the geometry of both the Euler and Navier–Stokes equations." [Gibbon 2008]

2. Differential Geometry: An Overview

Configuration and Phase Spaces

- Manifold M – looks locally like $\mathbb{R}^m$ but globally may have more structure i.e. $\mathbb{S}^m$ or $\mathbb{T}^m$.
- Configuration space/Background – m-dimensional manifold M with (local) coordinates $x^1, x^2, \cdots x^m$ w.r.t some basis $\{e_i\}_{i=1}^m$, with points x representing positions.
- A particle at a point $x \in M$ has an m-dimensional vector space of possible momenta with coordinates $q_1, q_2 \cdots q_m$ in some basis $\{\tilde{e}^i\}_{i=1}^m$.
- Phase space T^*M – $2m$-dimensional manifold, with (local) coordinates $x^1 \cdots x^m, q_1 \cdots q_m$, representing all possible combinations of positions and momenta.

Phase Space as the Cotangent Bundle

- Tangent space T_xM – Vector space of all tangent vectors to a point $x \in M$. Has a basis $\partial_{x^1}, \cdots \partial_{x^m}$.
- Cotangent space T_x^*M – Dual vector space to T_xM. Has a basis $\mathrm{d}x^1 \cdots \mathrm{d}x^m$, satisfying $\mathrm{d}x^i(\partial_{x^j}) = \delta^i_j$ (act on vector fields at x).
- Cotangent bundle T^*M – The $2m$-dimensional manifold consisting of (disjoint union of) M and its cotangent spaces.
- Differential 1-Form on M – $C^\infty(M)$ linear combination of $\mathrm{d}x^i$ and elements of T^*M i.e. $\alpha = \alpha_i(x^1, \cdots x^m)\,\mathrm{d}x^i \in T^*M := \Omega^1(M)$

Wedge Product

- Wedge Product $\wedge$ – A skew-symmetric bilinear operator on forms, so $\alpha \wedge \beta = -\beta \wedge \alpha$ for 1-forms α, β.
- The wedge product of two 1-forms on M defines a 2-form i.e. $\alpha_i \mathrm{d}x^i \wedge \beta_j \mathrm{d}x^j = (\alpha_i \beta_j)\mathrm{d}x^i \wedge \mathrm{d}x^j \in \Omega^2(M)$.
- The wedge product of k differential 1-forms is a k-form. More generally, a k-form is a totally skew-symmetric k-linear operator on vector fields: $\gamma = \gamma_{i_1,\cdots i_k}\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$
- The wedge product of a k-form $\tilde{\alpha}$ with an ℓ-form $\tilde{\beta}$ is a $(k+\ell)$-form satisfying: $\tilde{\alpha} \wedge \tilde{\beta} = (-1)^{k\ell}\tilde{\beta} \wedge \tilde{\alpha}$.

Exterior Derivative

- Exterior Derivative – Operator $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ taking k-forms to $(k+1)$-forms.
- Exterior derivative of a function:
  $\mathrm{d}(f) = (\partial_i f)\mathrm{d}x^i$ (think total derivative).
- Exterior derivative of a 1-form:
  $\mathrm{d}(\alpha_i \mathrm{d}x^i) = (\mathrm{d}\alpha_i) \wedge \mathrm{d}x^i = (\partial_j \alpha_i)\mathrm{d}x^j \wedge \mathrm{d}x^i = \frac{1}{2}(\partial_j \alpha_i - \partial_i \alpha_j)\mathrm{d}x^j \wedge \mathrm{d}x^i$.
- Note: $\mathrm{d}^2 = 0$ i.e.

$$\begin{aligned}\mathrm{d}^2(f) &= \mathrm{d}(\partial_i f \mathrm{d}x^i) = \mathrm{d}(\partial_i f) \wedge \mathrm{d}x^i \\ &= (\partial_j \partial_i f)\mathrm{d}x^j \wedge \mathrm{d}x^i = 0\end{aligned}$$

3. Monge–Ampère Geometry and 2D Navier–Stokes Flows

Monge–Ampère Structures

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A Monge–Ampère structure [Banos 2002] is a pair (ω, α) of differential forms on $T^*\mathbb{R}^m$, where:

$\omega \in \Omega^2(T^*\mathbb{R}^m)$ is a symplectic differential 2-form

- ω is anti-symmetric and bilinear
- ω is closed, i.e. $\mathrm{d}\omega \equiv 0$
- ω is non-degenerate, i.e. $\omega(X, \cdot) \equiv 0$ iff the vector-field $X \equiv 0$.

The canonical choice is

$$\omega = \mathrm{d}q_i \wedge \mathrm{d}x^i = \begin{pmatrix} 0_m & -I_m \\ I_m & 0_m \end{pmatrix}$$

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$\alpha \in \Omega^m(T^*\mathbb{R}^m)$ is an ω-effective differential m-form

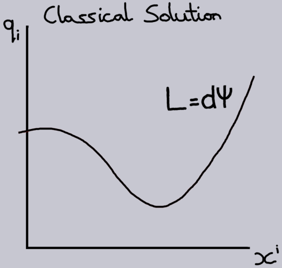
- α and ω are skew-orthogonal, i.e. $\alpha \wedge \omega = 0$,
- α is called the MA form.

Recovering Equations and Solutions

- Consider m-dimensional submanifolds $L = \{x^1, \cdots x^m, \partial_1\psi \cdots \partial_m\psi\} := \mathrm{d}\psi$ in $T^*\mathbb{R}^m$ for $\psi \in C^\infty(\mathbb{R}^m)$.
- Impose the pull-back (restriction) of ω and α to L vanish, i.e.

$$\begin{aligned}\omega|_L &= (\mathrm{d}q_i \wedge \mathrm{d}x^i)|_L \\ &= \mathrm{d}(\partial_i\psi) \wedge \mathrm{d}x^i = (\partial_j\partial_i\psi)\mathrm{d}x^j \wedge \mathrm{d}x^i = 0\end{aligned}$$

- The condition $\alpha|_L = 0$ then gives our MA equation, with classical solutions ψ. [Lychagin 1979]



Monge–Ampère Equations in Two Dimensions

The ω-effective MA forms for 2D background $(m = 2)$ are

$$\alpha = A\,\mathrm{d}q_1 \wedge \mathrm{d}x^2 + B\left(\mathrm{d}x^1 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}x^2\right) + C\,\mathrm{d}x^1 \wedge \mathrm{d}q_2 + D\,\mathrm{d}q_1 \wedge \mathrm{d}q_2 + E\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\,.$$

Imposing that their pull-back to $L = \{x^1, x^2, \partial_1\psi, \partial_2\psi\}$ vanish yields $(x^1 = x,\, x^2 = y)$

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0\,,$$

This correspondence is a bijection (unique MA form in ω-effective class).

Monge–Ampère Equations in Two Dimensions

- Recall canonical symplectic form has $\omega|_L = 0$.
- Hence $(\alpha + \omega)|_L = \alpha|_L + \omega|_L = \alpha|_L$, so α and $\alpha + \omega$ give the same MA equation.
- As ω is non-degenerate, $\omega \wedge \omega \neq 0$, but recall $\alpha \wedge \omega = 0$ by effectiveness.
- Hence $(\alpha + \omega) \wedge \omega = \alpha \wedge \omega + \omega \wedge \omega \neq 0$, so $(\alpha + \omega)$ not ω-effective.

Monge–Ampère Equations in Two Dimensions

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- Pfaffian f is defined by $\alpha \wedge \alpha =: f\omega \wedge \omega$.
- In fact, $f = AC - B^2 - DE$ which, on L, is the determinant of the linearisation matrix for our PDE.
- Hence, the MA equation $\alpha|_L = 0$ is

  elliptic $\Leftrightarrow f > 0$.
  hyperbolic $\Leftrightarrow f < 0$.
  parabolic $\Leftrightarrow f = 0$.

The Lychagin–Rubtsov Theorem and Equivalence

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- Define a $J : \mathfrak{X}(T^*\mathbb{R}^2) \to \mathfrak{X}(T^*\mathbb{R}^2)$ by

$$\frac{1}{\sqrt{|f|}}\alpha(\cdot\,,\cdot) =: \omega(J\,\cdot\,,\cdot) \qquad \left(J = \frac{1}{\sqrt{|f|}}\omega^{-1}\alpha \text{ as matrices}\right),$$

  for which $f \lessgtr 0 \Leftrightarrow J^2 = \pm 1$. [Lychagin et al. 1993]

- The Lychagin–Rubtsov theorem states t.f.a.e:
  - d$(\frac{1}{\sqrt{|f|}}\alpha) = 0$.
  - $\alpha|_L = 0$ is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$.
  - J is integrable.

Geometry of the 2D Poisson Equation

One can recover the pressure equation

$$\frac{\Delta p}{2} = \left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) \; ,$$

by choosing the MA form [Roulstone et al. 2009]

$$\alpha = \mathrm{d}q_1 \wedge \mathrm{d}q_2 - \frac{\Delta p}{2}\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

$$= \begin{pmatrix} 0 & -\frac{\Delta p}{2} & 0 & 0 \\ \frac{\Delta p}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Geometry of the 2D Poisson Equation

- The Pfaffian is then $f = \frac{\Delta p}{2}$.
- Geometric justification for Poisson equation being

$$\textit{elliptic} \Leftrightarrow \Delta p > 0.$$
$$\textit{hyperbolic} \Leftrightarrow \Delta p < 0.$$
$$\textit{parabolic} \Leftrightarrow \Delta p = 0.$$

- Also, the Poisson equation for pressure is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$ if and only if Δp is constant.

Extension to Riemannian Manifold

- On a Riemannian manifold (M, g), the approach is broadly the same:

$$\Delta p + R_{ij} v^i v^j = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} \,.$$

- Schematically take

$$\mathrm{d}q_i \to \mathrm{d}q_i - \mathrm{d}x^j {\Gamma_{ij}}^k q_k.$$
$$I \to g.$$
$$f = \tfrac{1}{2}\Delta p \to f = \tfrac{1}{2}(\Delta p + R^{ij} q_i q_j).$$

- Geometric justification for Weiss criterion for equation type still applies on a manifold, e.g. $\mathbb{S}^2$.

Navier–Stokes equations in spherical geometry describe ocean/atmosphere dynamics (Joshua Stevens - NASA Earth Observatory)

Velocity and Divergence

- Rather than working with the stream function, consider working with velocity directly. Set $\tilde{L} = \{x^1, x^2, v_1(x), v_2(x)\}$.
- $\alpha|_{\tilde{L}} = 0$ gives the Poisson equation for pressure in terms of vorticity and strain, but $\omega|_{\tilde{L}} = 0$ now implies the vorticity vanishes.
- Use a different symplectic form

$$\varpi = \mathrm{d}q_1 \wedge \mathrm{d}x^2 - \mathrm{d}q_2 \wedge \mathrm{d}x^1$$

whose pullback to $\tilde{L}$ is equivalent to $\nabla \cdot v = 0$.

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- In higher dimensions, the divergence-free and Poisson equations form a Jacobi system with solution $v(x)$, which can be studied using higher-symplectic geometry in this way. [Cantrijn et al. 2009]

4. The Lychagin–Rubtsov Metric and its Pullback

Lychagin–Rubtsov Metric

- For a choice of (non-degenerate, effective) 2-form $K \in \Omega^2(T^*\mathbb{R}^2)$, can define a symmetric, bilinear form

$$g(\cdot\,,\cdot) := -K(J\,\cdot\,,\cdot) \qquad (g := -KJ \text{ as matrices})\,,$$

called the Lychagin–Rubtsov (LR) metric, [Roulstone et al. 2001].

- There exists a choice of K s.t. the metric in $\mathrm{d}x^i$, $\mathrm{d}q_i$ basis is

$$g = \begin{pmatrix} fI & 0 \\ 0 & I \end{pmatrix}$$

with signature dictated by the sign of f.

Lychagin–Rubtsov Metric for the Poisson Equation

The LR metric on $T^*\mathbb{R}^2$ given by

$$g = \begin{pmatrix} \frac{\Delta p}{2} I & 0 \\ 0 & I \end{pmatrix}$$

is

$$\text{Riemannian} \Leftrightarrow \Delta p > 0.$$
$$\text{Kleinian} \Leftrightarrow \Delta p < 0.$$
$$\text{Degenerate} \Leftrightarrow \Delta p = 0.$$

N.B. These degeneracies correspond to singularities of the scalar curvature — they persist under coordinate changes.

Pull-back Metric for the Poisson Equation

- The pull-back of the LR metric to $L = \mathrm{d}\psi$ is

$$g|_L = \zeta \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix}$$

  where $\zeta = \Delta\psi$ is vorticity.

- Degenerate when $\zeta = 0$ or $\Delta p = 0$.
  Riemannian when $\Delta p > 0$.
  Kleinian when $\Delta p < 0$.

- Degeneracy when $\zeta = 0$ not always curvature singularity.

Geometry of the 2D Poisson Equation

Δp	> 0	< 0	$= 0$
Dominance	Vorticity	Strain	None
$\alpha\|_L = 0$	Elliptic	Hyperbolic	Parabolic
f	> 0	< 0	$= 0$
J^2	-1	1	Singular
g	Riemannian $(4, 0)$	Kleinian $(2, 2)$	Degenerate
$g\|_L$	Riemannian $(2, 0)$	Kleinian $(1, 1)$*	Degenerate

*Except when $\zeta = 0$, in which case it is degenerate
(Agrees with Larchevêque that sign of ζ is constant when $\Delta p > 0$).

5. Conclusion and Outlook

Summary

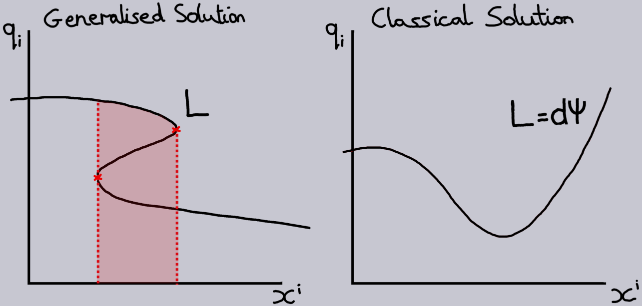
- We introduced MA structures and their associated geometry as a tool for studying MA PDEs.
- We applied this tool to the pressure equation in 2D and showed that the signatures of the LR metric and its pull-back act as diagnostics for equation type and the dominance of vorticity and strain.
- We briefly discussed generalisations to higher dimensions and manifolds with curvature, providing geometric validation for the Weiss criterion in these cases.

Outlook: Topology of Vortices

- For simply connected regions V of 2D flows on which $\Delta p > 0$ and with boundary given by a closed stream-line, all streamlines within V are also closed (and convex). [Larchevêque 1993]
- Gauss–Bonnet Theorem relates the mean curvature of the boundary of the 'vortex' to its topology, and the gradients of vorticity and strain encoded by the curvature of $g|_L$.
- No Gauss–Bonnet in 3D, but can relate K to the helicity and hence to the topology of knots (Jones' polynomial, Gauss linking number, etc) [Liu and Ricca 2012, Ricca and Moffatt 1992].

Outlook: Generalised Solutions

Generalised solution to a MA equation w.r.t. structure (ω, α) is any (Lagrangian) submanifold $L \hookrightarrow T^*\mathbb{R}^m$ s.t. $\dim(L) = m$, $\omega|_L = 0$, and $\alpha|_L = 0$.

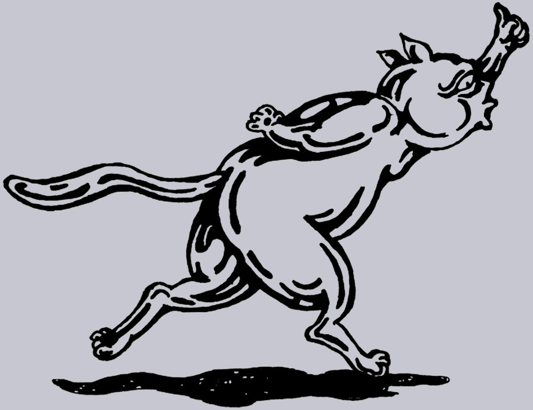


Can study discontinuity/lower regularity using multi-valued functions $\psi(x)$ [Vinogradov 1973, Kushner et al. 2007]

Outlook: Generalised Solutions

- What additional fluid behaviour is observed when allowing non-bijective projection into $\mathbb{R}^m$?
- In semi-geostrophic theory, these produce degeneracy of $g|_L$ and type change. Represents shockwaves in meteorological context. [D'Onofrio et al. 2023]
- Our geometry in 3D produces vortex tubes and lines for classical solutions — expect vortex sheets to be related to the singular locus of projections.

Thank you!



Any questions?