Monge–Ampère Geometry and the Navier–Stokes Equations

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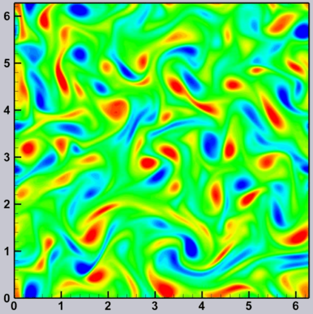
Types of Fluid Flow

Laminar Flow in a Sink
(Lucas Pereira - University of Stanford)

Wake Turbulence
(Ryoh Ishihara - via University of Illinois)

The Importance of Vortices

- Turbulent flows consist of complex interactions of vortex structures.
- In 2D, they combine as they evolve, forming stable coherent structures characterised by circulation/elliptic motion.
- In 3D, one finds knotted/linked tubes which accumulate at small scale. "sinews of turbulence." [Moffatt et al. 1994]



Vorticity of evolving 2d turbulence at early time (Andrey Ovsyannikov - Ecole Centrale de Lyon)

Outline of Talk

1. Introduction to Monge–Ampère Geometry
2. 2D Incompressible Navier–Stokes Flows
3. Higher Dimensions and Curved Backgrounds
4. Summary and Outlook

(Contact) Monge–Ampère Equations

- Are non-linear second-order PDEs which are linear w.r.t second order partial derivatives, up to a Hessian determinant.
- In two dimensions, they take the form

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0\,.$$

  where $A, B, \ldots E$ can depend on $x, y, \psi, \psi_x, \psi_y$ non-linearly.
- If $A, B, \ldots E$ do not depend on ψ, we have a symplectic Monge–Ampère (MA) equation.

Monge–Ampère Structures and Solutions

- The triple $(T^*\mathbb{R}^m, \omega, \alpha)$ with
  - ☞ $\omega \in \Omega^2(T^*\mathbb{R}^m)$ symplectic, e.g. $\omega = \mathrm{d}q_i \wedge \mathrm{d}x^i$,
  - ☞ $\alpha \in \Omega^m(T^*\mathbb{R}^m)$ is ω-effective, i.e. $\alpha \wedge \omega = 0$,

  is called a Monge–Ampère structure. [Banos 2002]

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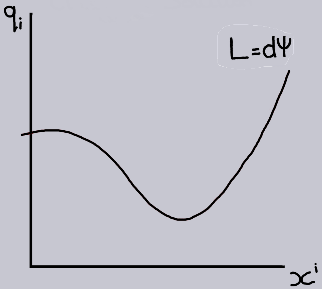
- A generalised solution to a MA equation, w.r.t. a MA structure, is a submanifold $\iota : L \hookrightarrow T^*\mathbb{R}^m$ s.t.
  - ☞ L is Lagrangian, i.e. $\dim(L) = m$ and $\iota^*\omega = 0$.
  - ☞ α vanishes on L, i.e. $\iota^*\alpha = 0$.

  [Kushner et al. 2007]

Recovering PDEs and Classical Solutions

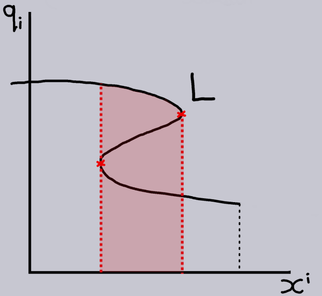
- Consider L with coordinates $(x^i, \partial_i\psi)$ for some $\psi \in \mathscr{C}^\infty(\mathbb{R}^m)$.
- This is Lagrangian, trivially satisfying $\iota^*\omega = (\mathrm{d}\psi)^*\omega = 0$.
- The constraint $\iota^*\alpha = (\mathrm{d}\psi)^*\alpha = 0$ is the corresponding MA equation, with classical solution ψ. [Lychagin 1979]
- The projection $\pi : L \to \mathbb{R}^m$ is a diffeomorphism.



More on Generalised Solutions

Pathologies of a generalised solution L:

- When $\pi : L \to \mathbb{R}^m$ is not surjective (ψ is not defined on the whole domain).
- When $\pi : L \to \mathbb{R}^m$ is not injective (ψ is a multivalued solution). [Vinogradov 1973]
- When $\pi : L \to \mathbb{R}^m$ is not immersive — Arnold's Singularities.



Monge–Ampère Equations in Two Dimensions

The general MA equation in two dimensions

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0$$

is (uniquely) given by the pull-back of the ω-effective MA form

$$\begin{aligned}\alpha = {} & A\,\mathrm{d}q_1 \wedge \mathrm{d}x^2 + B\left(\mathrm{d}x^1 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}x^2\right) \\ & + C\,\mathrm{d}x^1 \wedge \mathrm{d}q_2 + D\,\mathrm{d}q_1 \wedge \mathrm{d}q_2 + E\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\end{aligned}$$

to $L = \mathrm{d}\psi$.

Monge–Ampère Equations in Two Dimensions

- The Pfaffian is defined by $\alpha \wedge \alpha =: f\omega \wedge \omega$ where $f = AC - B^2 - DE$ is the determinant of the linearisation matrix for our PDE.

- Hence, the MA equation $\iota^*\alpha = 0$ is

$$
\begin{aligned}
&\textit{elliptic} \Leftrightarrow f > 0.\\
&\textit{hyperbolic} \Leftrightarrow f < 0.\\
&\textit{parabolic} \Leftrightarrow f = 0.
\end{aligned}
$$

The Lychagin–Rubtsov Theorem and Equivalence

- One can define an almost (para-)complex structure on $T^*\mathbb{R}^2$

$$\frac{\alpha}{\sqrt{|f|}} =: J \lrcorner\, \omega \,,$$

for which $f \lessgtr 0 \Leftrightarrow J^2 = \pm 1$. [Lychagin et al. 1993]

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- The Lychagin–Rubtsov theorem states t.f.a.e:
  - ☞ $\mathrm{d}(J \lrcorner \omega) = 0$.
  - ☞ $(\mathrm{d}\psi)^*\alpha = 0$ is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$.
  - ☞ J is integrable.

The Lychagin–Rubtsov Metric

- Choosing $K \in \Omega^2(T^*\mathbb{R}^2)$, we can define a symmetric, bilinear form $\hat{g}(X, Y) = K(X, JY)$ — Lychagin–Rubtsov (LR) metric. [Roulstone et al. 2001]
- There exists a choice of K s.t. the metric in (x^i, q_i) coordinates is

$$\hat{g} = \begin{pmatrix} fI & 0 \\ 0 & I \end{pmatrix}$$

with signature dictated by the sign of f.

Interlude

- MA equations are second-order PDEs whose second-order nonlinearity is a Hessian determinant.
- Encoded by a symplectic form ω and an ω-effective form α.
- Solutions are given by submanifolds of $T^*\mathbb{R}^m$ which are isotropic w.r.t. ω and α.
- Type of a 2D MA equation is indicated by the Pfaffian f.
- There exists an almost (para-)Hermitian structure $(J, K, \hat{g})$ on $T^*\mathbb{R}^m$, where $f \neq 0$.

Pressure, Vorticity, and Strain

- Navier–Stokes equations on $\mathbb{R}^m$ with coordinates x^i are

$$\frac{\partial v^i}{\partial t} = -v^j \nabla_j v^i - \nabla^i p + \nu \Delta v^i \,.$$

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$$\frac{\partial v^i}{\partial t} = -v^j \nabla_j v^i - \nabla^i p + \nu \Delta v^i \,.$$

- Applying the incompressibility constraint $\nabla \cdot v = 0$ one finds

$$\Delta p = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} \quad \text{with} \quad \zeta_{ij} = \nabla_{[j} v_{i]} \ \text{ and } \ S_{ij} = \nabla_{(i} v_{j)} \,.$$

- Vorticity term dominates $\Leftrightarrow$ $\Delta p > 0$.
  Strain term dominates $\Leftrightarrow$ $\Delta p < 0$.

The Pressure Equation

(This) equation for the pressure is by no means fully understood and locally holds the key to the formation of vortex structures through the sign of the Laplacian of the pressure. In this relation... may lie a deeper knowledge of the geometry of both the Euler and Navier–Stokes equations." [Gibbon 2008]

Pressure Equation in Two Dimensions

- In 2D, one has a stream function $v_1 = -\psi_y$ and $v_2 = \psi_x$.
- Pressure equation is a MA equation for the stream function

$$\frac{\Delta p}{2} = \left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) \, .$$

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- *Vorticity dominates* $\Leftrightarrow$ $\Delta p > 0$ $\Leftrightarrow$ *Elliptic equation.*
  Strain dominates $\Leftrightarrow$ $\Delta p < 0$ $\Leftrightarrow$ *Hyperbolic equation.*
  No dominance $\Leftrightarrow$ $\Delta p = 0$ $\Leftrightarrow$ *Parabolic equation.*
  [Weiss 1991, Larchevêque 1993]

Geometry of the 2D Poisson Equation

One can recover the pressure equation

$$\frac{\Delta p}{2} = \left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right)$$

by choosing the MA form [Roulstone et al. 2009]

$$\alpha = \mathrm{d}q_1 \wedge \mathrm{d}q_2 - f\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\,,$$

with Pfaffian given by

$$f = \frac{\Delta p(x,y)}{2}\,.$$

Almost (para-)Complex Structure

Recall that when $\Delta p =: 2f \neq 0$, we have an almost (para-)complex structure J defined via

$$\frac{\alpha}{\sqrt{|f|}} =: J \lrcorner \omega \,,$$

By Lychagin–Rubtsov Theorem, it follows

$$\frac{\Delta p}{2} = (\psi_{xx}\psi_{yy} - \psi_{xy}^2)$$

is locally equivalent to $\Delta \psi = 0$ or $\Box \psi = 0$ if and only if Δp is constant.

Associated Metrics

The LR metric on $T^*\mathbb{R}^2$ given by

$$\hat{g} = \begin{pmatrix} \frac{\Delta p}{2} I & 0 \\ 0 & I \end{pmatrix}$$

is

$$\text{Riemannian} \Leftrightarrow \Delta p > 0.$$
$$\text{Kleinian} \Leftrightarrow \Delta p < 0.$$
$$\text{Degenerate} \Leftrightarrow \Delta p = 0.$$

N.B. These degeneracies correspond to singularities of the scalar curvature — they persist under coordinate changes.

Associated Metrics

- The pullback metric on (ι, L) given by a classical solution $\mathrm{d}\psi$ is

$$(\mathrm{d}\psi)^* \hat{g} = \zeta \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix}$$

where $\zeta = \Delta\psi$.

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  where $\zeta = \Delta\psi$.

- Degenerate when $\zeta = 0$ or $\Delta p = 0$.
  Riemannian when $\Delta p > 0$.
  Kleinian when $\Delta p < 0$.

- Degeneracy when $\zeta = 0$ not always curvature singularity.

Geometry of the 2D Poisson Equation

Δp	> 0	< 0	$= 0$
Dominance	Vorticity	Strain	None
$(\mathrm{d}\psi)^*\alpha = 0$	Elliptic	Hyperbolic	Parabolic
f	> 0	< 0	$= 0$
J^2	-1	1	Singular
$\hat{g}$	Riemannian $(4,0)$	Kleinian $(2,2)$	Degenerate
$(\mathrm{d}\psi)^*\hat{g}$	Riemannian $(2,0)$	Kleinian $(1,1)^*$	Degenerate

*Except when $\zeta = 0$, in which case it is degenerate.

A Key Observation in 2D

- The divergence-free constraint $\nabla \cdot v = 0$ is encoded by the vanishing of the pull-back of

$$\begin{aligned} \varpi &= \mathrm{d}q_i \wedge \star\, \mathrm{d}x^i \\ &= \mathrm{d}q_1 \wedge \mathrm{d}x^2 - \mathrm{d}q_2 \wedge \mathrm{d}x^1 \end{aligned}$$

to submanifolds L with coordinates $(x^i, v_i(x))$.

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  to submanifolds L with coordinates $(x^i, v_i(x))$.

- ϖ is a symplectic form, so could be used in place of ω.

- Superficial change in 2D, upon noting $v = \star \mathrm{d}\psi$, but crucial in higher dimensions when the pressure equation is not MA (there is no stream function).

Higher Symplectic Monge–Ampère Problems

- Closed and non-degenerate $\varpi \in \Omega^{k+1}(T^*\mathbb{R}^m)$ are called k-plectic forms. [Cantrijn et al. 2009]
- Consider structures of form $(T^*\mathbb{R}^m, \varpi, \alpha)$ where ϖ is $(m-1)$-plectic.
- Generalised solutions are submanifolds $\iota : L \hookrightarrow T^*\mathbb{R}^m$ satisfying $\iota^*\varpi = 0$ and $\iota^*\alpha = 0$.
- We focus on L with coordinates $(x^i, v_i(x))$, diffeomorphic to $\mathbb{R}^m$, in lieu of classical solutions.

Pressure Equation in m-Dimensions

- Pulling back the following structure to classical solutions gives the divergence-free and pressure equations ($2f = \Delta p$):

$$\varpi = \mathrm{d}q_i \wedge \star\,\mathrm{d}x^i$$
$$\alpha = \tfrac{1}{2}\mathrm{d}q_i \wedge \mathrm{d}q_j \wedge \star\,(\mathrm{d}x^i \wedge \mathrm{d}x^j) - f\,\mathrm{vol}_m$$

- Such higher-symplectic MA structures seem to encode coupled vector equations.

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- For flows with symmetry, we now have access to k-plectic reduction to simplify say, 3D problems to 2D ones. [Blacker 2021]

Associated Metrics

- Can again define a metric on $T^*\mathbb{R}^m$ of the following form

$$\hat{g} = \begin{pmatrix} fI_m & 0 \\ 0 & I_m \end{pmatrix} .$$

- For $A_{ij} = \nabla_j v_i$, the pullback metric is

$$(\iota^* \hat{g})_{ij} = A^k{}_i A_{kj} - \tfrac{1}{2}\delta_{ij} A_{kl} A^{lk} .$$

- In general, signature change of $\iota^*\hat{g}$ does not coincide with sign change in f — more complicated relationship.

Extension to Riemannian Manifold

- On a Riemannian manifold (M, g), the approach is broadly the same:

$$\Delta p + R_{ij} v^i v^j = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} \, .$$

- Schematically take

$$\mathrm{d}q_i \to \mathrm{d}q_i - \mathrm{d}x^j \Gamma_{ij}{}^k q_k .$$
$$I \to g .$$
$$f = \tfrac{1}{2}\Delta p \to f = \tfrac{1}{2}(\Delta p + R^{ij} q_i q_j) .$$

- Geometric justification for Weiss criterion for equation type still applies on a manifold, e.g. $\mathbb{S}^2$.

Navier–Stokes equations in spherical geometry describe ocean/atmosphere dynamics (Joshua Stevens - NASA Earth Observatory)

Summary

- We introduced MA structures and their associated geometry as a tool for studying MA PDEs.
- We applied this tool to the pressure equation in 2D and showed that the signatures of the LR metric and its pull-back act as diagnostics for equation type and the dominance of vorticity and strain.
- We briefly discussed generalisations to higher dimensions and manifolds with curvature, providing geometric validation for the Weiss criterion in these cases.

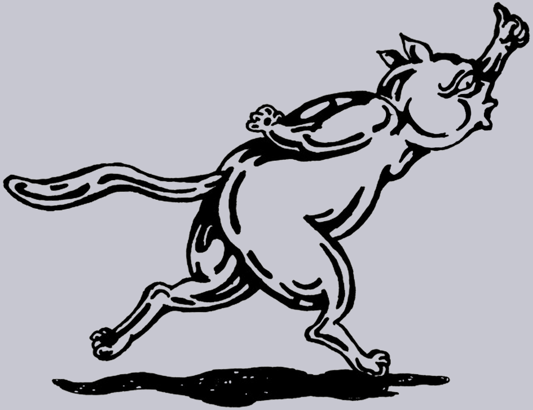
Outlook

- What additional behaviour is observed when allowing non-immersive projections?
- In semi-geostrophic theory, these produce degeneracy of $\iota^* \hat{g}$ and type change. [D'Onofrio et al. 2023]
- Geometry of classical solutions is related to elliptic vortices, tubes, and lines — expect vortex sheets to be related to the singular locus of projections.

Outlook

- One can classify 2D and 3D MA equations as locally equivalent to Laplace/ Wave type equations. [Banos 2003]
- Can extended generalised geometry provide a similar classification for higher-symplectic equations/ Jacobi systems? [Banos 2007]
- Our LR metric is closely related to the scaled Sasakian metrics, whose associated structures have been studied in detail. [Gezer et al. 2014]

Thank you!



Any questions?