# Monge-Ampère Geometry and the Navier–Stokes Equations

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# Outline

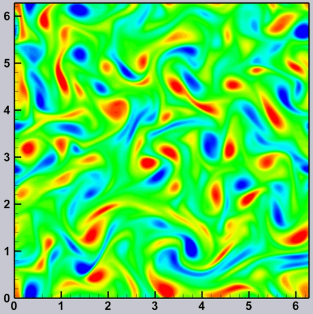
1. Vortices and Incompressible Fluids
2. Monge–Ampère Structures
3. Geometry of Fluid Flows in 2D
4. Fluid Flows in Higher Dimensions
5. Additional Results
6. Outlook

# The Importance of Vortices





- Turbulent flows consist of complex interactions of vortex structures.
- In 2D, they combine as they evolve, forming stable coherent structures characterised by circulation/elliptic motion.
- In 3D, one finds knotted/linked tubes which accumulate at small scale. "sinews of turbulence." [Moffatt et al. 1994]




Vorticity of evolving 2d turbulence at early time (Andrey Ovsyannikov - Ecole Centrale de Lyon)

# Pressure, Vorticity, and Strain





- Navier–Stokes equations on $\mathbb{R}^m$ with coordinates $x^i$ are

$$\frac{\partial v^i}{\partial t} = -v^j \nabla_j v^i - \nabla^i p + \nu \Delta v^i \,.$$

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$$\frac{\partial v^i}{\partial t} = -v^j \nabla_j v^i - \nabla^i p + \nu \Delta v^i \,.$$

- Applying the incompressibility constraint $\nabla \cdot v = 0$ one finds

$$\Delta p = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} \quad \text{with} \quad \zeta_{ij} = \nabla_{[j}v_{i]} \ \text{and} \ S_{ij} = \nabla_{(i}v_{j)} \,.$$

- Vorticity term dominates $\Leftrightarrow$ $\Delta p > 0$.
  Strain term dominates $\Leftrightarrow$ $\Delta p < 0$.

# The Pressure Equation





*"(This) equation for the pressure is by no means fully understood and locally holds the key to the formation of vortex structures through the sign of the Laplacian of the pressure. In this relation... may lie a deeper knowledge of the geometry of both the Euler and Navier–Stokes equations."* [Gibbon 2008]

# Pressure Equation in Two Dimensions

- In 2d, one has a stream function $v_1 = -\psi_y$ and $v_2 = \psi_x$.
- Pressure equation is a Monge–Ampère equation for the stream function [Larchevêque 1993]

$$\frac{\Delta p}{2} = \left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) .$$

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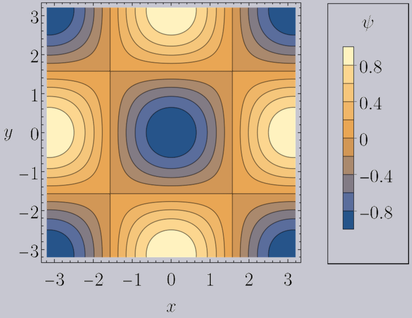
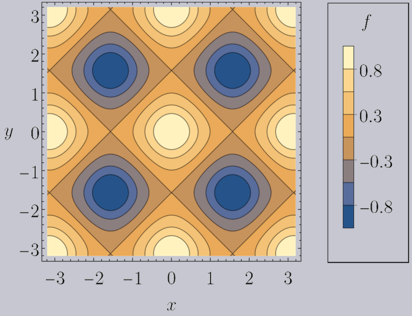
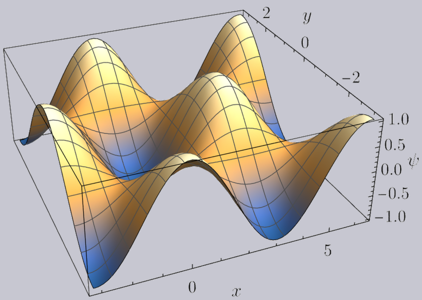
- *Vorticity dominates* $\Leftrightarrow$ $\Delta p > 0$ $\Leftrightarrow$ *Elliptic equation.*
  *Strain dominates* $\Leftrightarrow$ $\Delta p < 0$ $\Leftrightarrow$ *Hyperbolic equation.*
  *No dominance* $\Leftrightarrow$ $\Delta p = 0$ $\Leftrightarrow$ *Parabolic equation.*
  [Weiss 1991]

# Taylor–Green Vortex




$$\psi(x,y) = -\cos(x)\cos(y)$$
$$= -\zeta(x,y)$$



# Monge–Ampère Equations





- Are non-linear second-order PDEs which are linear w.r.t second order partial derivatives, up to a Hessian determinant.
- In two dimensions, they take the form

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0\,.$$

- Can recast them in terms of differential forms on phase space — Monge–Ampère (MA) structures.

# Monge–Ampère Structures and Solutions





- Monge–Ampère structures are $(T^*\mathbb{R}^m, \omega, \alpha)$ with
  - $\omega \in \Omega^2(T^*\mathbb{R}^m)$ symplectic, i.e. $\omega = \mathrm{d}q_i \wedge \mathrm{d}x^i$.
  - $\alpha \in \Omega^m(T^*\mathbb{R}^m)$ satisfying $\alpha \wedge \omega = 0$.

  [Banos 2002]

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  [Banos 2002]

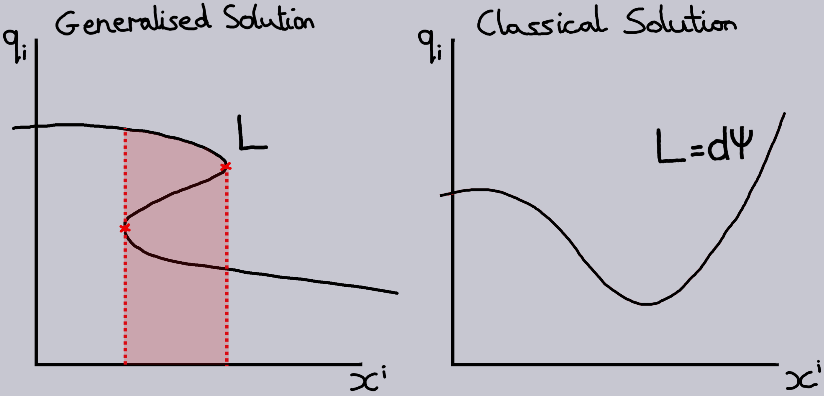
- A submanifold $\iota : L \hookrightarrow T^*\mathbb{R}^m$ is a generalised solution to a MA equation, w.r.t. a MA structure, if
  - ☞ $L$ is Lagrangian, i.e. $\dim(L) = m$ and $\iota^*\omega = 0$.
  - ☞ $\alpha$ vanishes on $L$, i.e. $\iota^*\alpha = 0$.

  [Kushner et al. 2007]

# Recovering Classical Solutions



If $L$ has coordinates $(x^i, \partial_i \psi)$, then $\iota^* \alpha = (d\psi)^* \alpha = 0$ is the corresponding MA equation, with $\psi \in \mathcal{C}^\infty(\mathbb{R}^m)$ a classical solution. [Lychagin 1979]



# Monge–Ampère Equations in Two Dimensions





- The general MA equation in two dimensions is given by the effective MA form

$$\begin{aligned}\alpha = {} & A\,\mathrm{d}q_1 \wedge \mathrm{d}x^2 + B\left(\mathrm{d}x^1 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}x^2\right) \\ & + C\,\mathrm{d}x^1 \wedge \mathrm{d}q_2 + D\,\mathrm{d}q_1 \wedge \mathrm{d}q_2 + E\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\end{aligned}$$

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- Pfaffian $\alpha \wedge \alpha =: f\omega \wedge \omega$ is given by $f = AC - B^2 - DE$.
- The Monge–Ampère equation $\iota^*\alpha = 0$ is

$$\textit{elliptic} \Leftrightarrow f > 0.$$
$$\textit{hyperbolic} \Leftrightarrow f < 0.$$
$$\textit{parabolic} \Leftrightarrow f = 0.$$

# Monge–Ampère Equations in Two Dimensions





- One can define an almost (para-)complex structure on $T^*\mathbb{R}^2$

$$\frac{\alpha}{\sqrt{|f|}} =: J \lrcorner\, \omega \,,$$

for which $f \lessgtr 0 \Leftrightarrow J^2 = \pm 1$. [Lychagin et al. 1993]

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- The Lychagin–Rubtsov theorem states t.f.a.e:
  - ☞ $\mathrm{d}(J \lrcorner \omega) = 0$.
  - ☞ $(\mathrm{d}\psi)^*\alpha = 0$ is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$.
  - ☞ $J$ is integrable.

# Almost (para-)Hermitian Metric

- Choosing $K \in \Omega^2(T^*\mathbb{R}^2)$, we can define a symmetric, bilinear form $\hat{g}(X, Y) = K(X, JY)$ — Lychagin–Rubtsov (LR) metric.
- Earlier works first fix $\hat{g}$ in terms of $(\omega, \alpha)$, corresponding to one choice of $K$. [Roulstone et al. 2001]

# Almost (para-)Hermitian Metric





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- Earlier works first fix $\hat{g}$ in terms of $(\omega, \alpha)$, corresponding to one choice of $K$. [Roulstone et al. 2001]
- We instead make a choice of $K$, s.t. the metric in $(x^i, q_i)$ coordinates is

$$\hat{g} = \begin{pmatrix} fI & 0 \\ 0 & I \end{pmatrix}$$

with signature dictated by the sign of $f$.

# Geometry of the 2D Poisson Equation

One can recover the pressure equation

$$\frac{\Delta p}{2} = \left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right)$$

by choosing the MA form [Roulstone et al. 2009]

$$\alpha = \mathrm{d}q_1 \wedge \mathrm{d}q_2 - f\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\,,$$

with Pfaffian given by

$$f = \frac{\Delta p(x,y)}{2}\,.$$

# Geometry of the 2D Poisson Equation





The LR metric given by

$$\hat{g} = \begin{pmatrix} \frac{\Delta p}{2} I & 0 \\ 0 & I \end{pmatrix}$$

is

$$\text{Riemannian} \Leftrightarrow \Delta p > 0.$$
$$\text{Kleinian} \Leftrightarrow \Delta p < 0.$$
$$\text{Degenerate} \Leftrightarrow \Delta p = 0.$$

N.B. These degeneracies correspond to singularities of the scalar curvature — they persist under coordinate changes.

# Geometry of the 2D Poisson Equation





- The pullback metric on $(\iota, L)$ given by a classical solution $\mathrm{d}\psi$ is

$$(\mathrm{d}\psi)^* \hat{g} = \zeta \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix}$$

where $\zeta = \Delta\psi$.

# Geometry of the 2D Poisson Equation





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$$(\mathrm{d}\psi)^*\hat{g} = \zeta \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix}$$

  where $\zeta = \Delta\psi$.

- Degenerate when $\zeta = 0$ or $\Delta p = 0$.
  Riemannian when $\Delta p > 0$.
  Kleinian when $\Delta p < 0$.

- Degeneracy when $\zeta = 0$ not always curvature singularity.

# Geometry of the 2D Poisson Equation

| | | | |
|---|---|---|---|
| $\Delta p$ | $> 0$ | $< 0$ | $= 0$ |
| Dominance | Vorticity | Strain | None |
| $(\mathrm{d}\psi)^*\alpha = 0$ | Elliptic | Hyperbolic | Parabolic |
| $f$ | $> 0$ | $< 0$ | $= 0$ |
| $J^2$ | $-1$ | $1$ | Singular |
| $\hat{g}$ | Riemannian $(4,0)$ | Kleinian $(2,2)$ | Degenerate |
| $(\mathrm{d}\psi)^*\hat{g}$ | Riemannian $(2,0)$ | Kleinian $(1,1)^*$ | Degenerate |

*Except when $\zeta = 0$, in which case it is degenerate.

# Interlude

Key questions:

- Pressure equation in 3D is not Monge–Ampère, but is similar. What is the correct framework for studying such equations?
- Can we reduce higher dimensional problems with symmetry to simpler 2D problems?
- Given that we are now in a geometric framework, do any additional features emerge from studying the Navier–Stokes equations on a Riemannian Manifold?

# A 2D Observation

- The above discussion also works with the symplectic form $\varpi = \mathrm{d}q_i \wedge \star\, \mathrm{d}x^i$.
- The pressure equation is given by $\iota^* \alpha = 0$ when a solution $(\iota, L)$ has coordinates $(x^i, v_i(x))$.

# A 2D Observation





- The above discussion also works with the symplectic form $\varpi = \mathrm{d}q_i \wedge \star\, \mathrm{d}x^i$.
- The pressure equation is given by $\iota^*\alpha = 0$ when a solution $(\iota, L)$ has coordinates $(x^i, v_i(x))$.
- As a bonus, this choice encodes incompressibility:

$$\iota^*\varpi = \nabla^i v_i = 0\,.$$

- The MA equation for pressure is recovered by noting $v = \star\, \mathrm{d}\psi$.

# Higher Symplectic Monge–Ampère Problems





- Closed and non-degenerate $\varpi \in \Omega^{k+1}(T^*\mathbb{R}^m)$ are called $k$-plectic forms. [Cantrijn et al. 2009]
- Consider structures of form $(T^*\mathbb{R}^m, \varpi, \alpha)$ for where $\varpi$ is now $(m-1)$-plectic.

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- Consider structures of form $(T^*\mathbb{R}^m, \varpi, \alpha)$ for where $\varpi$ is now $(m-1)$-plectic.
- We shall call submanifolds $\iota : L \hookrightarrow T^*\mathbb{R}^m$ generalised solutions if $\iota^*\varpi = 0$ and $\iota^*\alpha = 0$.
- We focus on $(\iota, L)$ with coordinates $(x^i, v_i(x))$, such that $\dim(L) = m$, in lieu of classical solutions.

# Geometry of Higher Dimensional Flows

- For $\varpi \in \Omega^m(T^*\mathbb{R}^m)$, one makes the choice

$$\varpi = \mathrm{d}q_i \wedge \star\, \mathrm{d}x^i \,.$$

Pulling this back to $L$ with coordinates $(x^i, v_i)$ gives incompressibility.

# Geometry of Higher Dimensional Flows





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  Pulling this back to $L$ with coordinates $(x^i, v_i)$ gives incompressibility.

- For $\alpha \in \Omega^m(T^*\mathbb{R}^m)$, one makes the choice [Roulstone et al. 2009]

  $$\alpha = \tfrac{1}{2}\mathrm{d}q_i \wedge \mathrm{d}q_j \wedge \star(\mathrm{d}x^i \wedge \mathrm{d}x^j) - f\,\mathrm{vol}_m$$

  for $2f = \Delta p$, which pulls back to the pressure equation.

# Geometry of Higher Dimensional Flows

- Our structure appears to encode coupled (vector) equations.
- Can again define the LR metric on $T^*\mathbb{R}^m$ in terms of an endomorphism $J$ [Banos 2002, Hitchin 2000]

$$\hat{g} = \begin{pmatrix} f I_m & 0 \\ 0 & I_m \end{pmatrix} .$$

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- For $A_{ij} = \nabla_j v_i$, the pullback metric is

$$(\iota^* \hat{g})_{ij} = A^k{}_i A_{kj} - \tfrac{1}{2} \delta_{ij} A_{kl} A^{lk} .$$

# Three Dimensional Flows With Symmetry

- The class of 2.5D flows take the form [Ohkitani et al. 2000]:

$$(\dot{x}, \dot{y}, \dot{z}) := (v_1(x, y, t), v_2(x, y, t), z\gamma(x, y, t) + W(x, y, t)) \ .$$

- For Burgers' vortex $W \equiv 0, \gamma = \gamma(t)$, one can symplectically reduce to a flow in 2d satisfying modified pressure/compressibility equations. [Banos et al. 2016]

- One may perform a reduction when $\gamma \equiv 0$ or $W(x, y) = c\gamma(x, y)$, as there is a 1D symmetry.

# Three Dimensional Flows With Symmetry





- Standard symplectic reduction yields equations for $v_i$ and an LR metric on the reduced space $\cong T^*\mathbb{R}^2$.
- Also have access to higher symplectic reduction, which yields equations in terms of $v_3$ and a stream function $\psi$. [Blacker 2021]
- The case $\gamma \equiv 0$ can also be extended to background manifolds with metric

$$g = g_2 + \mathrm{e}^{-2\varphi}\mathrm{d}x^3 \otimes \mathrm{d}x^3$$

i.e. cylindrical or spherical domains.

# Extension to Riemannian Manifold





- On a Riemannian manifold $(M, g)$, the approach is broadly the same,

$$\Delta p + R_{ij} v^i v^j = \zeta_{ij} \zeta^{ij} - S_{ij} S^{ij} \, .$$

- Schematically take

$$\mathrm{d}q_i \to \mathrm{d}q_i - \mathrm{d}x^j {\Gamma_{ij}}^k q_k .$$
$$I_m \to g .$$
$$f = \tfrac{1}{2}\Delta p \to f = \tfrac{1}{2}(\Delta p + R^{ij} q_i q_j) .$$

Navier–Stokes equations in spherical geometry describe ocean/atmosphere dynamics (Joshua Stevens - NASA Earth Observatory)

# Outlook

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- Can the different late-time behaviour of vortices in 2D and 3D be related to the topology of solutions $(\iota, L)$? i.e. via Maslov class?
- What additional behaviour is observed when allowing non-immersive projections? Related to degeneracy of $\iota^*\hat{g}$ and flow type change in semi-geostrophic theory. [D'Onofrio et al. 2023]
- For us, $\iota^*\hat{g}$ may degenerate due to coordinate choice. Instead, classifying curvature singularities may tell us more about type change and vortex formation.

# Outlook

- Classify 2D MA equations with integrable $J$ as locally equivalent to Laplace/wave equation.
- Can be extended to 3D MA equations under the added condition that $\hat{g}$ is flat. [Banos 2003]

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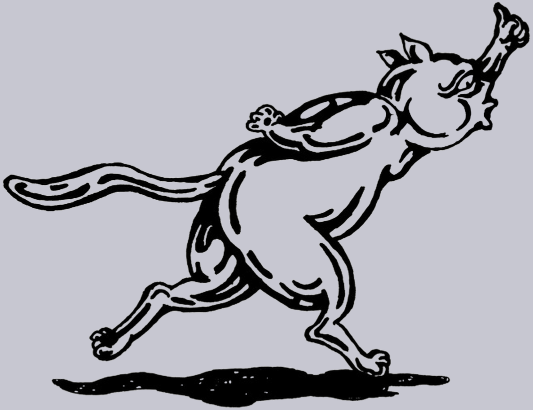
- Classify 2D MA equations with integrable $J$ as locally equivalent to Laplace/wave equation.
- Can be extended to 3D MA equations under the added condition that $\hat{g}$ is flat. [Banos 2003]
- Wish to classify higher symplectic equations to allow for similar local simplifications i.e. using generalised complex structures. [Banos 2007]
- Our LR metric is closely related to the scaled Sasakian metrics, whose associated structures have been studied in detail. [Gezer et al. 2014]

# Thank you!



# Any questions?