Monge–Ampère Geometry and Vortices

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Abstract

We present a covariant formulation of the incompressible Navier–Stokes equations on an arbitrary Riemannian manifold. We demonstrate that the associated Monge–Ampère equation for the pressure in two dimensions induces an almost(para)-Hermitian structure and that this structure may be generalised to higher dimensions. The signature and curvature of the resulting Lychagin–Rubtsov metric are proposed as diagnostic tools to determine the dominance of vorticity and strain in a fluid region and methods of deducing topological information about the flow are discussed. Symplectic and k-plectic reduction regimes are introduced as a method of simplifying three-dimensional incompressible flows with symmetry to a two-dimensional problem. Explicit examples of our constructions are provided in two and three dimensions. As this report is to be submitted for examination at the end of the confirmation period of a PhD programme, Appendix D catalogues the training modules completed over the previous 15 months, along with two extended abstract seminar reviews, against which 20 further hours of training are to be claimed.

Data and Licence Management

As a theoretical study, no practical data was collected or recorded to produce this report, hence no concerns about the ethical, safe storage/sharing of data arose. There are no immediate ethical concerns about the impact of the computations contained within this report. No additional research data beyond that presented and cited within this work is required to validate the findings herein. Citations to any any known relevant sources are provided on the following pages.

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1 Introduction

One of the enduring challenges of fluid mechanics is to understand the topology fluid flows. In particular, one wishes to describe the types of topological artefact that may be introduced by the presence of vortices. However, at present, there is no systematic method to glean such information from the underlying governing system of partial differential equations. Indeed, there is not even a universally applicable definition of a vortex. This report aims to collate and extend current research into how Monge–Ampère geometry and higher dimensional generalisations thereof may be able to provide further insight into these problems.

1.1 Larchevêque, Weiss, and Pressure criterion

Weiss [2] proposed, via dynamical and numerical arguments, a local criterion for identifying regions of elliptic/hyperbolic flow in two-dimensional fluids, modelled by the incompressible Euler equations. Said criterion considers the difference between the magnitude of the rate of strain squared and the square of the vorticity. It states that when the strain term dominates, the flow is hyperbolic, and when the vorticity term dominates, the flow is elliptical.

A geometric argument for the Weiss criterion was later provided by Larchevêque in [3, 4]; we present a brief summary of the approach here. We begin this construction from the incompressible Navier–Stokes flow equations, which are written in the familiar form

$$\frac{\partial v^i}{\partial t} = -v^j \partial_j v^i - \partial^i p + \nu \Delta v^i \tag{1.1.1a}$$

and

$$\partial_i v^i = 0 \ , \tag{1.1.1b}$$

where $\Delta := \partial^i \partial_i$ is the standard Euclidean Laplacian, ν is the viscosity, and we use Einstein summation convention, as we shall for the remainder of this report. We shall make the assumptions under which the Navier–Stokes equations take this form rigorous in the next section. By applying the divergence operator to equation (1.1.1a) and using (1.1.1b), we come to a Poisson equation for the pressure $p(x^1, x^2)$

$$\Delta p \;=\; -(\partial_i v_j)(\partial_j v_i) \;=\; \tfrac{1}{2}\zeta^2 - S_{ij}S^{ij}\,, \tag{1.1.1c}$$

in terms of the vorticity ζ and the strain-rate tensor S_{ij}. Further, Larchevêque noted that, when one uses the incompressibility constraint (1.1.1b) to write the velocity $v(x^1, x^2)$ in terms of a stream function $\psi(x^1, x^2)$, equation (1.1.1c) takes the form of a Monge–Ampère equation

$$\tfrac{1}{2}\Delta p = \partial_1^2\psi\partial_2^2\psi - (\partial_1\partial_2\psi)^2\,. \tag{1.1.2}$$

When the strain term dominates, the Laplacian of pressure is negative, and the Monge–Ampère equation is of hyperbolic type. Similarly, when the vorticity term dominates, the Laplacian of pressure is positive, and the Monge–Ampère equation is elliptic. Phrases of this form will become a sort of mantra throughout this report. Larchevêque and Weiss both observe that the sign of (1.1.1c) corresponds to the sign of the Gaussian curvature of the stream function, from which it follows that, given a simply connected open subset $V \subseteq \mathbb{R}^2$ on which vorticity dominates and which is bounded by a closed streamline, then the surface given by $z = \psi(x^1, x^2)$ is convex and all streamlines in V are closed contours.

Building on the work of [5, 6] in the class of approximations to the Navier–Stokes equations which are applicable to ocean-atmosphere dynamics (specifically, balanced models such as the semi-geostrophic and quasi-geostrophic equations), it was demonstrated in [7] that, for two-dimensional incompressible flows, the Monge–Ampère equation (1.1.2) can be associated to an almost-hyper-Kähler geometry. A key component of the aforementioned geometry for our analysis is the almost-hyper-Kähler metric , which we shall denote $\hat{g}$ and refer to as the Lychagin–Rubtsov metric. It is noted in [8] that, while the Poisson equation for the pressure does not take the form of a Monge–Ampère equation in the usual sense when considered in three dimensions, one may still relate it to a Monge–Ampère structure via the generalised complex geometry of Hitchin [9] and Banos [10]. Symplectic reductions from a specific class of three-dimensional examples into two dimensions via the Marsden–Weinstein reduction process were also addressed in some detail in [11]. Since we shall discuss each of these points in turn throughout the body of the report, we provide no further detail here.

We close this section by reiterating a quote of Gibbon [12] also used in [11]. They state that equations of the form (1.1.1c) "locally hold(s) the key to the formation of vortical structures through the sign."[1] In the next section, we shall introduce the covariant Navier–Stokes equations and a generalisation of equation (1.1.1c) to flows on an arbitrary Riemannian manifold. We also provide an outline of the report proper, which builds upon this generalisation.

[1] Referring here to the sign of the Euclidean Laplacian of the pressure.

1.2 Covariant Navier–Stokes Equations

Let the domain of our fluid flow be given by an m-dimensional Riemannian manifold M^m with metric $\mathring{g}$, which we shall refer to as the background metric. We will omit the superscript m where the dimension of the manifold is not pertinent, or is otherwise clear from the context. Denote the exterior derivative on M by d, the volume form on M by 'vol_M', and the Hodge star with respect to $\mathring{g}$ by $\star_{\mathring{g}}$. Further, for differential k-forms $\eta \in \Omega^k(M)$, define the norm square $|\eta|^2$ by $|\eta|^2 \mathrm{vol}_M := \eta \wedge \star_{\mathring{g}} \eta$ and the codifferential acting on η by $\mathring{\delta}\eta := (-1)^{m(k-1)+1} \star_{\mathring{g}} \mathrm{d} \star_{\mathring{g}} \eta$. The Hodge Laplacian is then given by $\mathring{\Delta}_{\mathrm{H}} := \mathring{\delta}\mathrm{d} + \mathrm{d}\mathring{\delta}$.

With regard to the fluid, let its pressure and viscosity respectively be given by the function $p \in \mathscr{C}^\infty(M)$ and the scalar $\nu \in \mathbb{R}$. The fluid flow is then described by the velocity (co-)vector field - a one-parameter family of differential one-forms $v \in \Omega^1(M)$, parametrised by parameter time $t \in \mathbb{R}$. For simplicity we make the following additional assumptions:

- The fluid is homogeneous, that is, its density is uniform in space and constant in time. Without loss of generality then, we set the density to be identically 1.
- The net external force field acting on the fluid is divergence-free for all time. This is represented by a family of differential one-forms $c(t) \in \Omega^1(M)$ satisfying $\mathring{\delta}c \equiv 0$.

We then impose that the flow satisfies the covariant Navier–Stokes equation corresponding to our assumptions:

$$\frac{\partial v}{\partial t} \; = \; -(-1)^m \star_{\mathring{g}} (v \wedge \star_{\mathring{g}} \mathrm{d}v) - \tfrac{1}{2}\mathrm{d}|v|^2 - \mathrm{d}p - \nu \mathring{\Delta}_{\mathrm{H}} v + c \; , \tag{1.2.1a}$$

supplemented with the incompressibility condition

$$\mathring{\delta} v \; = \; 0 \; . \tag{1.2.1b}$$

Hence, by applying the codifferential to equation (1.2.1a), we recover an equation for the pressure solely in terms of the velocity field and the background metric

$$\mathring{\Delta}_{\mathrm{H}} p \; = \; -|\mathrm{d}v|^2 + \star_{\mathring{g}}(v \wedge \star_{\mathring{g}} \mathring{\Delta}_{\mathrm{H}} v) - \tfrac{1}{2}\mathring{\Delta}_{\mathrm{H}}|v|^2 \; . \tag{1.2.1c}$$

It is also possible to generalise (1.2.1c) a net external force c which is not divergence free, which results in (1.2.1c) gaining a $-\mathring{\delta}c$ term on the left hand side. As this additional term will remain grouped with the Laplacian of the pressure in all subsequent computations, we continue to suppress external forces for the remainder of the report.

So that we may present the above equations in coordinate form, let M have coordinates x^i for $i, j \ldots = 1, \ldots m$. Let the Levi-Civita connection associated with $\mathring{g}_{ij}$ be given by $\mathring{\nabla}_i$, with Christoffel symbols $\mathring{\Gamma}_{ij}{}^k$. The components of the Riemann and Ricci curvature tensors are respectively denoted $\mathring{R}_{ijk}{}^{l} := \partial_i \mathring{\Gamma}_{jk}{}^{l} - \partial_j \mathring{\Gamma}_{ik}{}^{l} + \mathring{\Gamma}_{jk}{}^{m}\mathring{\Gamma}_{im}{}^{l} - \mathring{\Gamma}_{ik}{}^{m}\mathring{\Gamma}_{jm}{}^{l}$ and $\mathring{R}_{ij} := \mathring{R}_{kij}{}^{k}$. We shall lower and raise indices using $\mathring{g}_{ij}$ and its inverse $\mathring{g}^{ij}$. In computing the Hodge Laplacian in coordinates, it is useful to invoke the Weitzenböck formula for p-forms $\eta = \frac{1}{p!}\eta_{i_1\cdots i_p}\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}$

$$(\mathring{\Delta}_{\mathrm{H}}\eta)_{i_1\ldots i_p} \;=\; -\mathring{\Delta}_{\mathrm{B}}\eta_{i_1\ldots i_p} + \mathring{R}^{m}{}_{[i_1}\eta_{|m|i_2\cdots i_p]}\,, \tag{1.2.2}$$

where $\mathring{\Delta}_{\mathrm{B}} := \mathring{g}^{ij}\mathring{\nabla}_i\mathring{\nabla}_j = \mathring{\nabla}^i\mathring{\nabla}_i$ denotes the Beltrami Laplacian. Here and in the following, parentheses (respectively, square brackets) denote normalised symmetrisation (respectively, anti-symmetrisation) of the enclosed indices. With our newly introduced notation, the differential one-form v is presented as $v = v_i\mathrm{d}x^i$ with $v_i = v_i(t, x^1, \ldots, x^m)$ and similarly for c. Equations (1.2.1) then become

$$\frac{\partial v^i}{\partial t} \;=\; -v^j\mathring{\nabla}_j v^i - \partial^i p + \nu\mathring{\Delta}_{\mathrm{B}}v^i - \nu\mathring{R}^{ij}v_j + c^i\;, \tag{1.2.3a}$$

$$\mathring{\nabla}_i v^i \;=\; 0\;, \tag{1.2.3b}$$

$$2f := \mathring{\Delta}_{\mathrm{B}}p + v^iv^j\mathring{R}_{ij} \;=\; -(\mathring{\nabla}_i v_j)(\mathring{\nabla}^j v^i)\;. \tag{1.2.3c}$$

It should be apparent that upon fixing $M = \mathbb{R}^m$ (in particular $m = 2$) with the standard Euclidean metric $\mathring{g}_{ij} = \delta_{ij}$ given by the m-dimensional Kronecker delta, equations (1.2.3) simplify to the more familiar (1.1.1) introduced in Section 1.1.

We now define the vorticity two-form and the strain tensor respectively by

$$\zeta_{ij} \;:=\; \mathring{\nabla}_{[i}v_{j]} \;=\; \partial_{[i}v_{j]} \quad\text{and}\quad S_{ij} \;:=\; \mathring{\nabla}_{(i}v_{j)}\;. \tag{1.2.4}$$

It is clear, when presented in local coordinates, that the velocity vector field is a Killing vector field precisely when the strain tensor vanishes. Further, define the velocity gradient tensor (VGT) by

$$A_{ij} \;:=\; \mathring{\nabla}_j v_i \;=\; S_{ij} - \zeta_{ij}\,. \tag{1.2.5}$$

Recall that, for any matrix B, its decomposition $B = B_{\mathrm{sym}} + B_{\mathrm{anti}}$ with $B_{\mathrm{sym}} = \frac{1}{2}(B + B^T)$ and $B_{\mathrm{anti}} = \frac{1}{2}(B - B^T)$, into symmetric and antisymmetric parts, satisfies the following identity:

$$\mathrm{tr}\left[B_{\mathrm{sym}}B_{\mathrm{sym}}\right] - \mathrm{tr}\left[B_{\mathrm{anti}}^T B_{\mathrm{anti}}\right] \;=\; \mathrm{tr}\left[B^2\right]\;. \tag{1.2.6}$$

Hence, it is possible to rewrite equation (1.2.3c) in the more descriptive form

$$2f \;=\; \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} \;=\; -A_j{}^iA_i{}^j\,, \tag{1.2.7}$$

from which it is clear that the Laplacian of pressure of an incompressible fluid flow on an arbitrary Riemannian manifold depends on the vorticity and strain of the flow, as it does in the two-dimensional, Euclidean case (1.1.1c). However, the left hand side exhibits dependence on the

Ricci tensor of the underlying manifold, as well as the velocity without derivatives, which do not present themselves for flows on flat background.

Finally, as a result of the Poincaré lemma, it is always possible to locally solve, on an open and contractible set $U \subseteq M$, the incompressibility constraint (1.2.1b), using the Hodge dual of sections on U. Namely,

$$v = \star_{\mathring{g}} \mathrm{d}\psi \quad \text{for} \quad \psi \in \Omega^{m-2}(U) \quad \leftrightarrow \quad v^i = \frac{\sqrt{\det(\mathring{g})}}{(m-2)!} \varepsilon^{i_1 \cdots i_{m-1} i} \partial_{i_1} \psi_{i_2 \cdots i_{m-1}} , \tag{1.2.8}$$

where $\varepsilon_{i_1 \cdots i_m}$ is the Levi-Civita symbol with $\varepsilon_{1 \cdots m} = 1$; note that $\varepsilon^{1 \cdots m} = \frac{1}{\det(\mathring{g})} \varepsilon_{1 \cdots m}$. Upon subsituting this expression into (1.2.3c), we obtain a Monge–Ampère-type equation for the components of the differential form ψ. Generally, we may refer to $\psi \in \Omega^{m-2}(U)$ as the stream $(m-2)$-form. However, for $m = 2$, ψ is known, from the fluid dynamics, as the stream function - in this particular case we obtain a genuine Monge–Ampère equation. From this point onward, we assume that M has a 'good cover,' where finite intersections of open sets are contractible, in order to facilitate this. An equivalently acceptable notion is that spaces which are locally contractible.

In the remainder of this report, we extend the work of [7,8] to demonstrate that (1.2.3c) is a Monge–Ampère equation for two-dimensional incompressible flows on an arbitrary Riemannian background, and that there exists a related almost (para-)Hermitian metric on the associated cotangent bundle, see Chapter 2. Further, we investigate the properties of the Ricci curvature of the Lychagin–Rubtsov metric and its pullback, and discuss how topological information may be drawn from such quantities through the use of the local Gauß–Bonnet theorem. In Chapter 3 we demonstrate that there is a natural generalisation of this structure to fluid flows in higher dimensions, using the language of multi-symplectic geometry. The main body of the text ends with a discussion of the application of symplectic and k-plectic reductions to three-dimensional flows in Chapter 4, following on from the work of [11]. In doing so, we propose that, for three-dimensional flows with symmetry, it should be possible to consider some comparatively easy-to-plot two-dimensional visualisation.

We conclude in Chapter 5 by presenting an outlook of future endeavours, focusing on the potential use of generalised complex geometries to classify the k-plectic generalisation of Monge–Ampère equations, as well as examining the global properties of the submanifolds which represent solutions to Monge–Ampère equations in two dimensions.

2

Monge–Ampère Geometry and Two-Dimensional Fluids

In this chapter, we shall demonstrate that the Poisson equation for the pressure in an incompressible two-dimensional Navier–Stokes flow on a Riemannian manfold is in-fact a Monge–Ampère equation and present an associated Monge–Ampère structure. An almost (para-)complex form is found and the geometry induced by the related almost (para-)Hermitian metric is discussed. Before applying ourselves to our specific fluid dynamical problem, let us first recapitulate some of the fundamental definitions and results of Monge–Ampère theory.

2.1 Monge–Ampère Equations and their Associated Structures

We begin this chapter by recalling some of the key aspects of Monge–Ampère geometry, which are required for our analysis of (1.2.3c) as a Monge–Ampère equation in two spatial dimensions. We shall mostly follow the text book [13] and since we shall not deviate far from material relevant to our application, we direct the interested reader there for more details. First, we supply the ($2m$-dimensional) cotangent bundle T^*M of our m-dimensional manifold M with coordinates (x^i, q_i), where x^i are the local coordinates defined on M in the introduction, and q_i are the local fibre coordinates. Let ω denote a symplectic (closed, non-degenerate) form on T^*M; in particular, for our choice of coordinates, the canonical symplectic form is $\omega = \mathrm{d}q_i \wedge \mathrm{d}x^i$. The following structure can then be defined:

Definition 2.1.1 (Monge–Ampère Structure)
Following [14, 15], a differential m-form $\alpha \in \Omega^m(T^*M)$ is called ω-effective when $\omega \wedge \alpha = 0$. The pair (ω, α) is then called a Monge–Ampère structure on T^*M and α is referred to as the Monge–Ampère form.

Definition 2.1.2 (Embedded Submanifold)
Let L, N be smooth manifolds and $\iota : L \to N$ be a smooth (injective) immersion. The pair (L, ι) is called a smoothly immersed submanifold of N. If, in addition, the immersion is also a

topological homeomorphism onto its image $\iota(L) \subseteq N$ *(where* $\iota(L)$ *carries the subspace topology), then* (L, ι) *is called a smoothly embedded submanifold of* N*.*

From this point onwards, submanifolds are assumed to be smoothly embedded unless otherwise stated. As an abuse of terminology, $\iota(L)$ may also be referred to as a submanifold. Note also that the embedding ι is in-fact a diffeomorphism onto its image.

Definition 2.1.3 (Generalised Solutions of Monge–Ampère Structures)
A generalised solution of a Monge–Ampère structure (ω, α) *is a submanifold* $\iota : L \hookrightarrow T^*M$ *which is Lagrangian with respect to* ω*,* $(\dim(L) = \dim(M)$ *and* $\iota^*\omega = 0)$ *such that the additional constraint* $\iota^*\alpha = 0$ *holds.*

Now consider a (global) section $d\psi : M \to T^*M$ given by the differential of $\psi \in \mathscr{C}^\infty(M)$ and described by $x^i \to (x^i, \partial_i\psi)$. This defines a Lagrangian submanifold $\iota(L) := d\psi(M) \subset T^*M$ with respect to the standard symplectic form and the requirement that $\iota^*\alpha = 0$ then yields a Monge–Ampère partial differential equation

$$\Delta_\alpha(\psi) \;:=\; (d\psi)^*\alpha = \; 0\,, \tag{2.1.1}$$

for which ψ is a classical/regular solution. We will often abuse terminology and refer to $d\psi(M)$ as a regular solution of the Monge–Ampère structure. Note that, while classical solutions to a PDE are generally only required to be as differentiable as the maximum number p of derivatives taken, we restrict ourselves to considering smooth classical solutions above. This ensures that, for an m-dimensional manifold M, the Monge–Ampère operator $\Delta_\alpha : \mathscr{C}^\infty(M) \to \Omega^m(M) \cong \mathscr{C}^\infty(M)$ takes smooth functions to smooth functions. One approach to properly considering all classical solutions would be to treat Monge–Ampère operators of the form $\Delta_\alpha : \mathscr{C}^k(M) \to \Omega^m_{k-p}(M) \cong \mathscr{C}^{k-p}(M)$, with $k \geqslant p$, where $\Omega^m_{k-p}(M)$ denotes m-forms on M with only $(k-p)$-times differentiable coefficients. In the context of generalised solutions, the Lagrangian submanifold would then not be required to be smooth.

As v is only given by (1.2.8) on open, contractible subsets of M, it is not always necessary to consider Lagrangian submanifolds $\iota : L \hookrightarrow T^*M$ that are globally given by a section $d\psi : M \to T^*M$ satisfying $\iota(L) = d\psi(M)$. In light of this, we state the following definitions:

Definition 2.1.4 (Local Diffeomorphism)
A function $h : L \to M$ *is a local diffeomorphism if, for all* $y \in L$*, there exists some open neighbourhood* V *of* y *with* $V \subseteq L$*, such that* $h(V)$ *is open in* M *and the restriction* $h|_V$ *of* h *to* V *is a diffeomorphism onto its image. In this case, we may locally coordinatise* $h(V)$ *by the coordinates on* V *and vice-versa.*

Definition 2.1.5 (Locally-a-section Submanifolds)
A submanifold $\iota : L \hookrightarrow T^*M$ *is called locally-a-section if, for all* $y \in L$ *there exists* $V_y \subseteq L$ *an open neighbourhood of* y*,* $U_y \subseteq M$ *open, and* $\psi \in \mathscr{C}^\infty(U_y)$ *such that* $\iota(V_y) = d\psi(U_y)$*.*

Banos [10] makes, without proof or precise definitions, the following claim, in order to establish locally-a-section submanifolds as a 'nice' subset of generalised solutions:

Proposition 2.1.6 (Locally-a-section and Local Diffeomorphisms)
Let the cotangent bundle T^*M be equipped with the standard symplectic form ω and let $\pi : T^*M \to M$ be the canonical projection. A Lagrangian submanifold $\iota : L \hookrightarrow T^*M$ with respect to ω is locally-a-section in the sense of Definition 2.1.5 if and only if the map $\pi|_L \coloneqq \pi \circ \iota : L \to M$ is a local diffeomorphism in the sense of Definition 2.1.4.

A proof of this result may be found in Appendix A. Hence, for locally-a-section Lagrangian submanifolds, there exists an open neighbourhood V_y of any point $y \in L$, such that $\iota(V_y) = \mathrm{d}\psi_y(U_y)$, for some open $U_y \subseteq M$ and $\psi_y \in \mathscr{C}^\infty(U_y)$ the stream function determining the velocity vector field on U_y. Additionally, V_y and U_y can be chosen such that they are diffeomorphic by the above proposition, in which case U_y and V_y can be given the same coordinates and $\pi|_L$ is locally the identity (on each V_y).

We close this section by making some observations regarding when the configuration space M is two-dimensional, to make the presentation in the next section more transparent. Observe that, when $m = 2$, generalised solutions are bi-Lagrangian, that is, Lagrangian with respect to both ω and α. Further, it is known that Monge–Ampère equations in two-dimensional flat space take the general form

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - (\psi_{xy})^2\right) + E = 0\,, \tag{2.1.2}$$

where we adopt the notation $x^1 = x$, $x^2 = y$ and the coefficients $A, \ldots, E$ are smooth functions of $x, y, \psi, \psi_x,$ and ψ_y, where subscripts indicate partial derivatives. Observe that the only non-linearity of second order derivatives is the determinant of a Hessian, with coefficient D. Equation (2.1.2) may be naturally extended to curved, two-dimensional domains by replacing partial derivatives with covariant ones, though some care has to be taken with the definition of the determinant of the Hessian, as we shall see when we consider the two-dimensional pressure equation in the next section.

2.2 Geometric Properties of Two-Dimensional Incompressible Fluid Flows

Let us now specialise to incompressible fluid flows (that is, where the flow is incompressible and the fluid homogeneous) in $m = 2$ dimensions. In this case, the components of the Riemann tensor are given in terms of the Ricci scalar $\mathring{R}$ as $\mathring{R}_{ijk}{}^{l} = \mathring{R}\mathring{g}_{k[i}\delta_{j]}{}^{l}$, so that the components of the Ricci tensor are simply $\mathring{R}_{ij} = \frac{\mathring{R}}{2}\mathring{g}_{ij}$. Furthermore, we denote by $\mathsf{Hess}(\psi)$ the Hessian of a function $\psi \in \mathscr{C}^\infty(M)$, which reads as $\mathsf{Hess}(\psi) = (\mathring{\nabla}_i\partial_j\psi) = (\mathring{\nabla}_j\partial_i\psi)$ in local coordinates. Then, (1.2.8) yields $v^i = -\sqrt{\det(\mathring{g})}\varepsilon^{ij}\partial_j\psi$ and so, the pressure constraint (1.2.1c) on an open, contractible neighbourhood $U \subseteq M$ becomes

$$\tfrac{1}{2}\mathring{\Delta}_{\mathrm{B}} p \;=\; \det\big(\mathring{g}^{-1}\mathsf{Hess}(\psi)\big) - \tfrac{\mathring{R}}{4}|\mathrm{d}\psi|^2 \quad\Longleftrightarrow\quad \tfrac{1}{2}\mathring{\nabla}^i\partial_i p \;=\; \det\Big(\mathring{\nabla}^i\partial_j\psi\Big) - \tfrac{\mathring{R}}{4}(\mathring{\nabla}^i\psi)(\partial_i\psi)\,, \tag{2.2.1}$$

where the Hessian matrix is pre-multiplied by the inverse metric $\mathring{g}^{-1}$ so that the determinant can be taken as it would on a matrix.[1] Hence, solving for the stream function allows us to locally describe the velocity vector field. Observe that, by comparison with (2.1.2), the equation (2.2.1) takes the form of a Monge–Ampère equation for the stream function ψ on some arbitrary Riemannian manifold.

Importantly, it is possible to find Monge–Ampère structures from which the pressure constraint (2.2.1) arises when considering, for example, generalised solutions defined via sections, as discussed around (2.1.1). Indeed, it is possible to check that the differential forms

$$\begin{aligned} \omega \;&:=\; \mathring{\nabla} q_i \wedge \mathrm{d}x^i \;=\; \mathrm{d}q_i \wedge \mathrm{d}x^i\;, \\ \alpha \;&:=\; \tfrac{\sqrt{\det(\mathring{g})}}{2}\Big[\varepsilon^{ij}\mathring{\nabla} q_i \wedge \mathring{\nabla} q_j - \underbrace{\big(\tfrac{1}{2}\mathring{\Delta}_{\mathrm{B}} p + \tfrac{\mathring{R}}{4}|q|^2\big)}_{=:\,\hat{f}}\varepsilon_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j\Big] \end{aligned} \tag{2.2.2}$$

on T^*M, where $\mathring{\nabla} q_i := \mathrm{d}q_i - \mathrm{d}x^j\mathring{\Gamma}_{ji}{}^k q_k$, form a Monge–Ampère structure on T^*M. Recall that $\frac{1}{2}\omega\wedge\omega$ is the Liouville volume form with respect to ω. Observing $\alpha\wedge\alpha = \hat{f}\mathring{\nabla} q_i \wedge \mathrm{d}x^i \wedge \mathring{\nabla} q_j \wedge \mathrm{d}x^j = \hat{f}\omega\wedge\omega$ then yields that α is non-degenerate if and only if $\hat{f} \neq 0$. Additionally, in Section 3.3, we shall show that α is closed. Furthermore, while $\iota^*\omega = 0$ is automatic on Lagrangian submanifolds $\iota : L \hookrightarrow T^*M$ which are locally-a-section, the condition $\iota^*\alpha = 0$ is equivalent to ψ satisfying the Monge–Ampère equation (2.2.1). Additionally, note that pulling back α via $q_i \to v_i = (\star_{\mathring{g}}\mathrm{d}\psi)_i$ again yields (2.2.1) for incompressible flows; this observation shall inform the alternative Monge–Ampère structure chosen in Section 3.2, which more naturally generalises to higher dimensions.

Next, following [15] and as discussed in Section B.1, we associate with the Monge–Ampère structure (2.2.2) an endomorphism $\hat{J}$ of the tangent bundle of T^*M defined by

$$\frac{\alpha}{\sqrt{|\hat{f}|}} \;=:\; \hat{J} \lrcorner\, \omega\,, \tag{2.2.3}$$

with $\hat{f}$ as in (2.2.2) and under the assumption that $\hat{f}$ does not vanish. By virtue of the results of [15], $\hat{J}^2 = -\operatorname{sgn}(\hat{f})$, hence $\hat{J}$ is an almost complex structure on T^*M when $\hat{f} > 0$ (in which case the Monge–Ampère equation (2.2.1) is elliptic) and an almost para-complex structure on T^*M when $\hat{f} < 0$ (in which case the Monge–Ampère equation (2.2.1) is hyperbolic). Furthermore, we

[1] A matrix-like quantity $A_i{}^j$ has determinant $\det(A) = \frac{1}{2}\det(\mathring{g})\epsilon^{i_1,i_2}A_{i_1}{}^{j_1}A_{i_2}{}^{j_2}\epsilon_{j_1,j_2}$. Of course, we could alternatively define an analogous determinant for $(0,2)$ tensors by raising the indices on the second Levi-Civita symbol, but our choice makes for a more intuitive reading.

can always find a differential two-form $\hat{K}$ which is of type $(1,1)$ with respect to $\hat{J}$,[1] such that $\hat{K}\wedge\omega = 0$, $\hat{K}\wedge(\hat{J}\lrcorner\,\omega) = 0$, and $\hat{K}\wedge\hat{K}\neq 0$. Explicitly, we may take

$$\hat{K} \;:=\; -\mathrm{sgn}(\hat{f})\sqrt{|\hat{f}|}\,\mathring{\nabla}q_i\wedge\star_{\mathring{g}}\mathrm{d}x^i\ . \tag{2.2.4}$$

where $\hat{f}$ is again assumed to be non-vanishing, such that $\hat{K}$ is well defined and satisfies the aforementioned constraints.

Since $\hat{K}(\hat{J}X,Y) = -\hat{K}(X,\hat{J}Y)$ for all $X,Y\in\mathfrak{X}(T^*M)$, we are naturally led to the almost (para-)Hermitian metric $\hat{g}(X,Y) := \hat{K}(X,\hat{J}Y)$ on T^*M for all $X,Y\in\mathfrak{X}(T^*M)$, which is explicitly given by

$$\hat{g} \;=\; \tfrac{1}{2}\hat{f}\mathring{g}_{ij}\mathrm{d}x^i\odot\mathrm{d}x^j + \tfrac{1}{2}\mathring{g}^{ij}\mathring{\nabla}q_i\odot\mathring{\nabla}q_j\ . \tag{2.2.5}$$

Evidently, for $\hat{f} > 0$, in the elliptic case, the metric $\hat{g}$ is Riemannian while for $\hat{f} < 0$, the hyperbolic case, it is Kleinian. Observe that, by the Lychagin–Rubtsov theorem [13, 15], the endomorphism (2.2.3) is integrable if and only if $\hat{f}$ is constant, in which case $\hat{J}$ is a (para-)complex structure. Additionally, (2.2.4) is promoted to a Kähler form if and only if it is closed, which occurs precisely when $\hat{f}$ is constant. It follows that $\hat{f}$ being constant is a necessary and sufficient condition for (2.2.5) to be a Kähler metric.

Note also that the vorticity two-form (1.2.4) can be written as

$$\zeta_{ij} \;=\; \tfrac{1}{2}\sqrt{\det(\mathring{g})}\,\varepsilon_{ij}\,\zeta \quad\text{with}\quad \zeta \;:=\; \mathring{\Delta}_{\mathrm{B}}\psi \quad\Longrightarrow\quad \zeta_{ij}\zeta^{ij} \;=\; \tfrac{1}{2}\zeta^2\ , \tag{2.2.6}$$

where ζ is referred to as the vorticity of a two-dimensional flow, in the spirit of (1.1.1c). Then, it can be seen that the pullback $g := \iota^*\hat{g}$ of (2.2.5) to a Lagrangian submanifold L described by a section $\mathrm{d}\psi$ is

$$g \;=\; \tfrac{1}{2}g_{ij}\mathrm{d}x^i\odot\mathrm{d}x^j \quad\text{with}\quad g_{ij} \;:=\; \zeta\mathring{\nabla}_i\partial_j\psi\ , \tag{2.2.7}$$

where we have substituted (2.2.6) and used that

$$f \;:=\; \iota^*\hat{f} \;=\; \tfrac{1}{2}\mathring{\Delta}_{\mathrm{B}}p + \tfrac{\mathring{R}}{4}|\mathrm{d}\psi|^2 \;=\; \det\Big(\mathring{\nabla}^i\partial_j\psi\Big)\,, \tag{2.2.8a}$$

by the pressure constraint (2.2.1), as well as

$$\det\Big(\mathring{\nabla}^k\partial_l\psi\Big)\,\mathring{g}_{ij} + \mathring{g}^{kl}(\mathring{\nabla}_i\partial_k\psi)(\mathring{\nabla}_j\partial_l\psi) \;=\; \mathring{\Delta}_{\mathrm{B}}\psi\mathring{\nabla}_i\partial_j\psi\,. \tag{2.2.8b}$$

Evidently, in regions where the vorticity vanishes, this metric vanishes as well. Note also that when both $\mathrm{tr}\big(\mathring{g}^{ik}g_{kj}\big) > 0$ and $\det\big(\mathring{g}^{ik}g_{kj}\big) > 0$, it follows that g is Riemannian. As $\mathrm{tr}\big(\mathring{g}^{ik}g_{kj}\big) = \zeta^2$, the former condition is always satisfied, while $\det\big(\mathring{g}^{ik}g_{kj}\big) = \zeta^2\det\Big(\mathring{\nabla}^i\partial_j\psi\Big) = \zeta^2 f$ implies that the latter is satisfied if and only if $f > 0$. Similarly, $f < 0$ implies that g is Kleinian. Hence,

[1] That is, $\hat{K}$ is an almost (para-)Hermitian form.

the signature of g is independent of the sign of the vorticity (2.2.6) and only depends on the sign of f. Furthermore, since $\mathring{\nabla}_i \partial_j \psi$ is symmetric and continuous in x^i for $\psi \in \mathscr{C}^\infty(M)$, it follows that, on simply connected regions of M with $f > 0$, $\zeta = \operatorname{tr}\left(\mathring{\nabla}^i \partial_j \psi\right)$ has fixed sign. Hence, the vorticity does not vanish where it dominates, that is, where g is Riemannian.

Upon comparing (1.2.7) and (2.2.8a), we observe that $f = \frac{1}{2}(\zeta_{ij}\zeta^{ij} - S_{ij}S^{ij})$. Hence, when $f > 0$ and the metric g is Riemannian, vorticity dominates, yet when $f < 0$, strain dominates. This is essentially a covariantisation of the pressure criterion for a vortex, as given in [3, 4] and discussed in Section 1.1. That is, f compensates for the effect of the curvature $\mathring{R}$ on the Laplacian of pressure when the background metric is not flat. Hence, we have found a geometric motivation for Weiss-like criterion to hold on a Riemannian manifold and, as we shall see in the next chapter, also in higher dimensions.

2.2.1 Curvature and Topology of Two-Dimensional Incompressible Fluid Flows

Let us now consider the curvatures associated with the Lychagin–Rubtsov metric and its pullback, in order to observe how the topological nature of flow regimes may depend on the vorticity and strain. We refer the reader to Appendix C for more detail. To begin, we quote a result from said appendix, specialised to the two dimensional case; the curvature scalar for the metric (2.2.5) is given by

$$\begin{aligned}\hat{R} \;=\; & \tfrac{1}{\hat{f}}\mathring{R} - \tfrac{1}{4\hat{f}}\mathring{R}_{ijk}{}^{l}\mathring{R}^{ijkm}q_k q_m - \hat{\Delta}_B \log\left(|\hat{f}|\right) \\ & - \mathring{g}_{ij}\left[\frac{\partial^2}{\partial q_i \partial q_j}\log\left(|\hat{f}|\right) - \frac{1}{2}\frac{\partial}{\partial q_i}\log\left(|\hat{f}|\right)\frac{\partial}{\partial q_j}\log\left(|\hat{f}|\right)\right],\end{aligned} \tag{2.2.9}$$

where $\hat{\Delta}_B$ is the Beltrami Laplacian with respect to $\hat{g}$, $\mathring{R}$ and $\mathring{R}_{ijk}{}^{l}$ are the Ricci scalar and Riemann tensor of $\mathring{g}$ respectively, and $\hat{f}$ is defined as above.

The curvature scalar for the pullback metric (2.2.7) requires a little extra in the way of notation to write down. In particular, we define

$$\psi_{i_1\cdots i_n} \;\coloneqq\; \mathring{\nabla}_{(i_1}\cdots\mathring{\nabla}_{i_{n-1}}\partial_{i_n)}\psi\,, \tag{2.2.10}$$

for $n \in \mathbb{N}$. A brief calculation shows that, for $n > 1$, $\psi_{i_1\cdots i_n}$ can be expressed in terms of the components of the strain tensor (1.2.4) and the vorticity (2.2.6) as

$$\psi_{i_1\cdots i_n} \;=\; -\sqrt{\det(\mathring{g})}\,\mathring{g}^{jk}\varepsilon_{j(i_1}\mathring{\nabla}_{i_2}\cdots\mathring{\nabla}_{i_{n-1}}S_{i_n)k} + \tfrac{1}{2}\mathring{g}_{(i_1 i_2}\mathring{\nabla}_{i_3}\cdots\mathring{\nabla}_{i_{n-1}}\partial_{i_n)}\zeta\,. \tag{2.2.11}$$

Then, using (2.2.6), write $g_{ij} = \zeta\tilde{g}_{ij}$ with $\tilde{g}_{ij} \coloneqq \psi_{ij}$ which, for $\zeta \neq 0$, gives the pullback metric g a conformal structure, with conformal factor $|\zeta|$. Hence, the Christoffel symbols $\Gamma_{ij}{}^{k}$ of g_{ij} take the form

$$\Gamma_{ij}{}^{k} \;=\; \tilde{\Gamma}_{ij}{}^{k} + \partial_{(i}\delta_{j)}{}^{k}\log(|\zeta|) - \tfrac{1}{2}\tilde{g}_{ij}\tilde{g}^{kl}\partial_l\log(|\zeta|)\;, \tag{2.2.12a}$$

where $\tilde{g}^{ij}$ denotes the inverse of the Hessian metric $\tilde{g}_{ij}$, and the $\tilde{\Gamma}_{ij}{}^k$ are its Christoffel symbols, which in turn are given by

$$\tilde{\Gamma}_{ij}{}^k \;=\; \mathring{\Gamma}_{ij}{}^k + \tfrac{1}{2}\Upsilon_{ijl}\tilde{g}^{lk} \quad \text{with} \quad \Upsilon_{ijk} \;:=\; \psi_{ijk} + \tfrac{4}{3}\psi_l \mathring{R}_{k(ij)}{}^l \;. \tag{2.2.12b}$$

Consequently, the curvature scalar R of g_{ij} is given by

$$R \;=\; \tfrac{1}{\zeta}\left\{\tilde{R} - \tfrac{1}{\sqrt{|\det(\tilde{g})|}}\partial_i\Big[\sqrt{|\det(\tilde{g})|}\,\tilde{g}^{ij}\partial_j\log(|\zeta|)\Big]\right\} \;, \tag{2.2.13a}$$

where $\tilde{R}$ is the curvature scalar for $\tilde{g}_{ij}$,

$$\begin{aligned}\tilde{R} \;=\; &\tfrac{1}{2}\tilde{g}^{ij}\mathring{g}_{ij}\mathring{R} - \tfrac{1}{4}\tilde{g}^{ij}\tilde{g}^{kl}\tilde{g}^{mn}(\Upsilon_{ijm}\Upsilon_{kln} - \Upsilon_{ikm}\Upsilon_{jln}) \\ &+ \tfrac{2}{3}\tilde{g}^{ij}\tilde{g}^{kl}\big[\psi_{mn}\big(\delta_i^m\mathring{R}_{j(kl)}{}^n - \delta_j^m\mathring{R}_{l(ik)}{}^n\big) + \psi_m\big(\mathring{\nabla}_i\mathring{R}_{j(kl)}{}^m - \mathring{\nabla}_j\mathring{R}_{l(ik)}{}^m\big)\big]\,.\end{aligned} \tag{2.2.13b}$$

Importantly, no fourth-order derivatives of the stream function appear, and in that sense, the curvature scalar of the pullback metric (2.2.7) is generated by gradients of vorticity and strain, see (2.2.11). Furthermore, ψ_i occurs without any further derivatives, hence the curvature scalar depends also on the components of velocity directly. Note also that the conformal structure of the metric (2.2.7) isolates a factor of $\frac{1}{\zeta}$ in (2.2.13a), suggesting that contours along which vorticity vanishes may present as curvature singularities in the geometry. However, this does not hold in general — see Section 2.3.2 for a counterexample, where vanishing vorticity only induces degeneracies in the pullback metric (2.2.7) and not in the curvature (2.2.13a). For further discussion on the singularity structures of the Lychagin–Rubtsov metric and its pullback, see [16] in the context of semi-geostrophic theory.

Consider some incompressible Navier–Stokes flow with M its Riemannian background manifold and $\iota : L \hookrightarrow T^*M$ a locally-a-section Lagrangian submanifold defined by $\iota^*\omega = 0$ and $\iota^*\alpha = 0$, that is, L is a (sufficiently nice) generalised solution of (2.2.2). Then there exist open subsets $U \subseteq M$, $V \subseteq L$ and a stream function $\psi \in \mathscr{C}^\infty(U)$, such that $\iota(V) = \mathrm{d}\psi(U)$. Then by Proposition 2.1.6, U and V can be chosen such that they are diffeomorphic. Suppose now that there exists a compact region $\Sigma \subseteq U$ on which $f > 0$. Then we may define the compact region $L_\Sigma \subseteq V \subseteq L$ by $\iota(L_\Sigma) := \mathrm{d}\psi(\Sigma)$. In particular, if $\iota(L)$ is described (globally) by a section $\mathrm{d}\psi : M \to T^*M$, then we can do this for any compact subset of $\Sigma \subseteq M$. It is now natural to consider the question of how we might use the local Gauß–Bonnet theorem (stated below, or see [17, Theorem 4.2] for details) to relate the geometry of L_Σ, as described by the curvature provided above, to its topology, as given by the Euler characteristic $\chi(L_\Sigma)$. For convenience, we provide the necessary theorem here:

Theorem 2.2.1 (Local Gauß–Bonnet Theorem)

Let Σ be a two-dimensional compact oriented manifold with Riemannian metric g. Suppose that

Σ has a boundary that is composed of piecewise simple regular closed arc-length parametrised curves γ_α, $\partial\Sigma = \bigcup_\alpha \gamma_\alpha$. Let R be the curvature scalar of the Levi-Civita connection of g, vol_Σ the volume form on Σ, and κ the geodesic curvature. Furthermore, let φ_β be the exterior angles at the non-smooth points of the boundary $\partial\Sigma$. Then, the Euler number $\chi(\Sigma)$ of Σ is given by

$$\tfrac{1}{2}\int_\Sigma \mathrm{vol}_\Sigma\, R + \sum_\alpha \int_{\gamma_\alpha} \mathrm{d}s\, \kappa(\gamma_\alpha(s)) + \sum_\beta \varphi_\beta \;=\; 2\pi\chi(\Sigma)\ . \tag{2.2.14}$$

Given an arc-length parametrised curve $\gamma : s \to (x^1(s), x^2(s))$, there are many ways of expressing the geodesic curvature κ at a given point $\gamma(s)$. One particular expression, attributed to Beltrami, is suitable for our purposes. When adapted to the notation adopted for the metric (2.2.7), Beltrami's formula takes the form

$$\kappa(\gamma(s)) \;=\; \sqrt{|\det(g(x(s)))|}\, \varepsilon_{ij}\, \dot{x}^i(s)\big[\ddot{x}^j(s) + \Gamma_{kl}{}^j(x(s))\dot{x}^k(s)\dot{x}^l(s)\big]\ , \tag{2.2.15}$$

where the superposed dots indicate derivatives with respect to the arc-length parameter s.

Let $\Sigma \subseteq U \subseteq M$ and $L_\Sigma \subseteq V \subseteq L$ be as above. Then, since U and V are chosen to be diffeomorphic, it follows $\chi(\Sigma) = \chi(L_\Sigma)$. Recall that we impose $f > 0$ on Σ, such that the metric (2.2.7) remains Riemannian on L_Σ. Furthermore, if such a Σ is bounded by a simple closed curve, such as a closed isovortical contour, or a closed stream line, then Σ is homeomorphic to a disc, so $\chi(L_\Sigma) = 1$. Hence, upon considering compact $\Sigma \subseteq U \subseteq M$, with $f > 0$ and boundary given by a simple, closed, arc-length parametrised curve γ, (2.2.14) reduces to

$$\int_\gamma \mathrm{d}s\, \kappa(\gamma(s)) \;=\; 2\pi - \tfrac{1}{2}\int_\Sigma \mathrm{vol}_\Sigma\, R\ , \tag{2.2.16}$$

That is, the mean curvature of the boundary is determined by the average curvature of the interior. Noting (2.2.13a), (2.2.13b), and (2.2.15), we remark that at a formal qualitative level, the local Gauß–Bonnet relation (2.2.16) is a statement to the effect that

$$\begin{aligned}\text{mean curvature of the boundary } &= \\ &= 2\pi - \text{mean gradients of vorticity and strain.}\end{aligned} \tag{2.2.17}$$

In this sense, when $f > 0$, we can use Monge–Ampère geometry to assign a topology to a 'vortex', with the vortex being described by L_Σ – the image of the gradient of the stream function. When the pullback metric (2.2.7) is Kleinian, the Gauß–Bonnet theorem can be extended to such cases, under certain conditions pertaining to the boundary ∂L_Σ — e.g. it should have no null segments — however, the link between topology as quantified by the Euler characteristic and the Gauß–Bonnet theorem becomes much more tenuous [18, 19].

2.3 Incompressible Fluids on the Euclidean Plane — Examples

In this section, we adopt the notation $x := x^1$ and $y := x^2$ and consider flows in $\mathbb{R}^2$ with background metric $\mathring{g}_{ij} = \delta_{ij}$, to verify consistency with known results in the most straightforward case. We shall present the simplified formulæ in this case as well as computing the curvatures of the Lychagin–Rubtsov metric and its pullback for the classical example of a two-dimensional Taylor–Green vortex.

2.3.1 Simplified Formulae in the Euclidean Case

For convenience, we summarise the relevant simplified formulæ before providing explicit computations. Evidently, $\mathring{R} = 0$ in this case, and so we find for $\hat{f}$ given in (2.2.2) and f given below (2.2.7) that

$$\hat{f} \;=\; \tfrac{1}{2}\Delta p \;=\; \partial_x^2\psi\partial_y^2\psi - (\partial_x\partial_y\psi)^2 \;=\; f \quad \text{with} \quad \Delta \;:=\; \partial_x^2 + \partial_y^2\;. \tag{2.3.1}$$

Hence, the metric (2.2.5) on $T^*\mathbb{R}^2$ takes the form

$$\hat{g} \;=\; \operatorname{diag}(f\delta_{ij}, \delta^{ij})\,, \tag{2.3.2}$$

with its signature dictated by the sign of f. This is singular if and only if $f = 0$ and the corresponding curvature scalar (2.2.9) (see also (C.2.13)) becomes

$$\hat{R} \;=\; \tfrac{1}{f^3}\big(\partial_x f\partial_x f + \partial_y f\partial_y f - f\Delta f\big)\,. \tag{2.3.3}$$

The vorticity (2.2.6) is simply $\zeta = \Delta\psi$ for the stream function $\psi = \psi(x, y)$ so the pullback metric (2.2.7) becomes

$$(g_{ij}) \;=\; \zeta\begin{pmatrix}\partial_x^2\psi & \partial_x\partial_y\psi\\ \partial_x\partial_y\psi & \partial_y^2\psi\end{pmatrix} \;=\; \frac{\zeta}{2}\begin{pmatrix}\zeta + 2S_{xy} & -2S_{xx}\\ -2S_{xx} & \zeta - 2S_{xy}\end{pmatrix}, \tag{2.3.4}$$

where $S_{xx} = -S_{yy}$ and S_{xy} are the components of strain tensor (1.2.4), describing a shearing deformation at an angle of $\frac{1}{2}\arctan\left(\frac{S_{xy}}{S_{xx}}\right)$, without overall dilation, since our flow is divergence free [20, 21]. The shearing deformation can be best described physically as the elliptic stretching of a circle of particles about each hyperbolic fixed point. Graphically, the shearing angle manifests as the angle of the asymptotes of the hyperbolæ to the coordinate axes, at these fixed points. Observe that g is singular when the vorticity vanishes, in addition to when the Hessian part of the metric is singular, that is, where $f = 0$ (see (2.2.8a)). Another invariant of the velocity-gradient tensor, the resultant deformation D_{R} [20], occurs in the expression for the eigenvalues of (2.2.7),

$$E_\pm \;=\; \tfrac{1}{2}\big(\zeta^2 \pm |\zeta|D_{\mathrm{R}}\big) \quad \text{with} \quad D_{\mathrm{R}}^2 \;:=\; 4(\partial_x\partial_y\psi)^2 + \big(\partial_x^2\psi - \partial_y^2\psi\big)^2\,. \tag{2.3.5}$$

Note that $D_{\mathrm{R}}^2 = \zeta^2 - 4f$ and so, the eigenvalues take the same sign for $f > 0$ and opposite sign for $f < 0$, provided they are both non-zero, as should be expected from the discussion following (2.2.8a). In particular, this tells us that the coordinate singularities depend precisely on the vorticity and the resultant deformation (and in turn the vorticity and f). Finally, the curvature scalars (2.2.13) reduce to

$$R = \frac{1}{\zeta}\left\{\tilde{R} - \frac{1}{\sqrt{|\det(\tilde{g})|}}\partial_i\big[\sqrt{|\det(\tilde{g})|}\,\tilde{g}^{ij}\partial_j \log(|\zeta|)\big]\right\}, \tag{2.3.6a}$$

where

$$(\tilde{g}^{ij}) = \frac{1}{f}\begin{pmatrix} \partial_y^2\psi & -\partial_x\partial_y\psi \\ -\partial_x\partial_y\psi & \partial_x^2\psi \end{pmatrix}, \tag{2.3.6b}$$

and

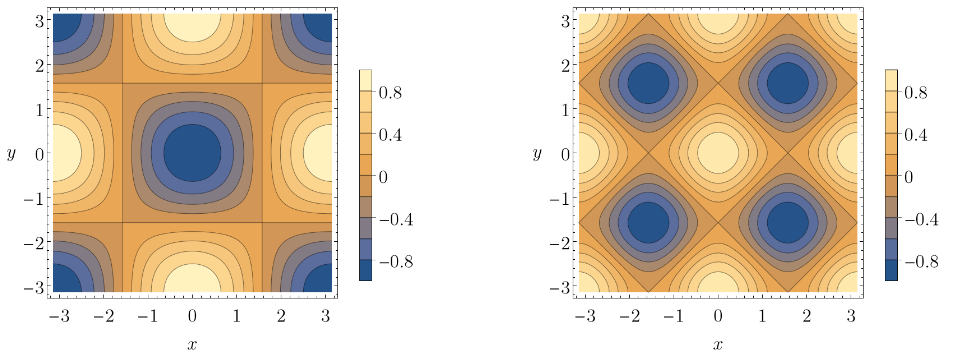
$$\tilde{R} = -\tfrac{1}{4}\tilde{g}^{ij}\tilde{g}^{kl}\tilde{g}^{mn}(\partial_i\partial_j\partial_m\psi\,\partial_k\partial_l\partial_n\psi - \partial_i\partial_k\partial_m\psi\,\partial_j\partial_l\partial_n\psi)\ . \tag{2.3.6c}$$

2.3.2 The Taylor–Green Vortex

The Taylor–Green vortex [22] is described by the stream function

$$\psi(x, y; t) := -F(t)\cos(ax)\cos(by)\,, \tag{2.3.7}$$

where F is a function of time t alone and $a, b \in \mathbb{R}$ are parameters. See Figure 2.3.1a. All plots for this example shall use parameters $a = b = 1$ and time t such that $F(t) = 1$.



(a) The streamlines of ψ, partitioning the domain into squares of side length π. The sign of ψ produces a check pattern.

(b) The contour plot for f, with positive/negative regions around elliptic/hyperbolic fixed points of the flow.

Figure 2.3.1: Plots of the iso-lines of (2.3.7) and (2.3.8). Observe that streamlines corresponding to values of sufficiently large magnitude are closed contours contained in regions of positive f, where vorticity dominates.

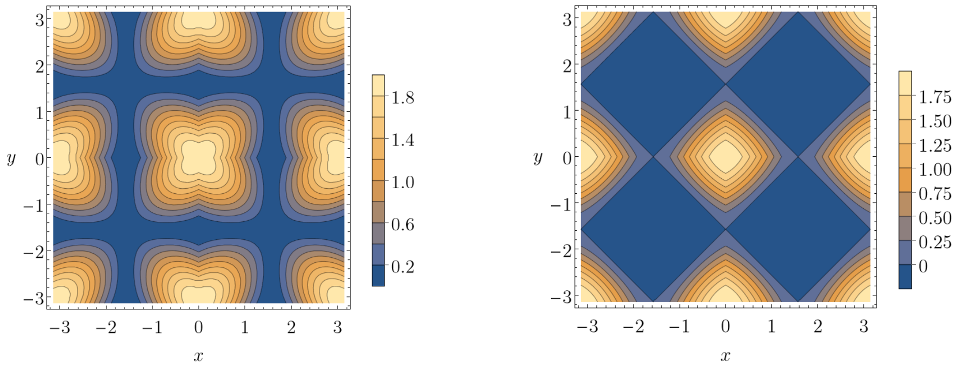
It follows that (2.3.1) is simply given by

$$f \;=\; \tfrac{1}{2}a^2b^2F^2[\cos(2ax)+\cos(2by)] \tag{2.3.8}$$

(see Figure 2.3.1b) and therefore, the curvature scalar (2.3.3) (see Figure 2.3.3a) is written

$$\hat{R} \;=\; \frac{8(a^2+b^2)[1+\cos(2ax)\cos(2by)]}{a^2b^2F^2[\cos(2ax)+\cos(2by)]^3}\,. \tag{2.3.9}$$

Consequently, when $(abF)^2 \neq 0$ and $\cos(2ax)+\cos(2by) > 0$, the metric (2.3.2) is Riemannian with a positive curvature scalar and vorticity dominates. Alternatively, taking $\cos(2ax)+\cos(2by) < 0$ yields that the metric is Kleinian with negative curvature scalar, and strain dominates. Observe that the signs of f and $\hat{R}$ coincide. Both the metric and curvature scalar are singular when $(abF)^2 = 0$ and along the lines $y = \pm\frac{a}{b}x + \frac{\pi}{2b}(2n+1)$ for all $n \in \mathbb{Z}$ (where $\cos(2ax)+\cos(2by) = 0$), corresponding to where $f = 0$.



(a) Contour plot for E_+, which is non-negative on the whole domain.

(b) Contour plot E_-, which is non-positive within the dark blue regions.

Figure 2.3.2: Plots of the eigenvalues (2.3.12) of the pullback metric (2.3.11).

Furthermore, the vorticity is given by

$$\zeta \;=\; (a^2+b^2)F\cos(ax)\cos(by) \;=\; -(a^2+b^2)\psi \tag{2.3.10}$$

and the pullback metric (2.3.4) becomes

$$(g_{ij}) \;=\; \frac{(a^2+b^2)F^2}{4}\begin{pmatrix} a^2[1+\cos(2ax)][1+\cos(2by)] & -ab\sin(2ax)\sin(2by) \\ -ab\sin(2ax)\sin(2by) & b^2[1+\cos(2ax)][1+\cos(2by)] \end{pmatrix}. \tag{2.3.11}$$

Its eigenvalues (2.3.5), as shown in Section 2.3.2 are given by

$$E_\pm \;=\; \frac{F^2(a^2+b^2)}{4}\left[2\big(a^2+b^2\big)\cos^2(ax)\cos^2(by) \pm |\cos(ax)\cos(by)|\sqrt{\tilde{E}}\,\right], \tag{2.3.12a}$$

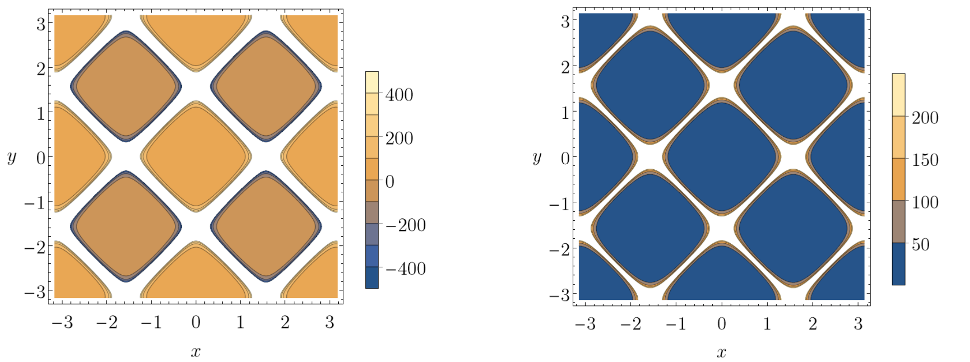
with

$$\tilde{E} \;:=\; \big(a^4 - 6a^2b^2 + b^4\big)[\cos(2ax) + \cos(2by)] + \big(a^2 + b^2\big)^2[1 + \cos(2ax)\cos(2by)] \;. \quad (2.3.12b)$$

Using (2.3.4), we can explicitly compute the components of strain to be $S_{xx} = S_{yy} = -Fab\sin(ax)\sin(by)$ and $S_{xy} = \frac{1}{2}F(a^2 - b^2)\cos(ax)\cos(by)$. It follows that the shear is parallel to the coordinate axes at the hyperbolic fixed points given by $\frac{\pi}{2}(2m+1, 2n+1)$ for all $m, n \in \mathbb{Z}$. The curvature scalars (2.3.6a) corresponding to (2.3.11) are

$$R \;=\; \frac{8}{F^2(a^2+b^2)[\cos(2ax)+\cos(2by)]^2} \quad \text{and} \quad \tilde{R} \;=\; 0 \;. \qquad (2.3.13)$$

Evidently, R is always positive, as shown in Figure 2.3.3b.



(a) Contour plot for the curvature scalar $\hat{R}$ on $T^*\mathbb{R}^2$. The signs of $\hat{R}$ and $\hat{f}$ agree.

(b) Contour plot for the curvature scalar R. This is positive on all $T^*\mathbb{R}^2$.

Figure 2.3.3: Contour plots of the curvatures (2.3.9) and (2.3.13) respectively. Note that both are singular on the curves $y = \pm x + \frac{\pi}{2}(2n+1)$ for all $n \in \mathbb{Z}$, along which $f = 0$.

Observe that E_+ is everywhere non-negative, so the signature of the metric (2.3.11) is determined by E_-. Note when $\cos(2ax) + \cos(2by) \gtrless 0$, $E_- \gtrless 0$, and the metric g is Riemannian/Kleinian and vorticity/strain dominates, despite the curvature R being everywhere positive. Also, $E_- = 0$ for $\cos(2ax) + \cos(2by) = 0$, which are hence coordinate singularities for g. In-fact, they are curvature singularities, see (2.3.13). Further, both eigenvalues (2.3.12) also vanish on the curves $x = \frac{\pi}{2a}(2n+1)$ and $y = \frac{\pi}{2b}(2n+1)$ for all $n \in \mathbb{Z}$, along which the stream function (2.3.7) and vorticity (2.3.10) also vanish. It follows that the metric (2.3.11) is degenerate along these curves, while the curvature scalar (2.3.13) is not singular, hence $x = \frac{\pi}{2a}(2n+1)$ and $y = \frac{\pi}{2b}(2n+1)$ are not curvature singularities of the metric and only arise due to the coordinate choice. Note that while the vorticity changes sign as the contours $x = \frac{\pi}{2a}(2n+1)$ or $y = \frac{\pi}{2b}(2n+1)$ are crossed, the metric (2.3.11) is Kleinian on both sides; this falls in line with Larchevêque's observations [4] that the vorticity has constant sign in Riemannian regions, where it dominates.

In summary, we observe that the contours described by $\cos(2ax) + \cos(2by) = 0$, along which f vanishes, are curvature singularities for the metrics $\hat{g}$ and g, while the contours $x = \frac{\pi}{2a}(2n+1)$ and $y = \frac{\pi}{2b}(2n+1)$, along which ζ vanishes, are only coordinate singularities for g. Note that the degeneracies of (2.2.5) and (2.2.7) which we have treated so far were local in the sense that they depend on $(x, y) \in \mathbb{R}^2$. Additionally, there are global degeneracies given by choices of a, b, and $F(t)$, which can be shown to arise as a result of more trivial flow scenarios. For example, setting $F \equiv 0$ or $a = b = 0$ fixes $\psi(x, y; t) = -F(t)$, in which case all components of the velocity vanish and the fluid is stationary. Alternatively, when only $a = 0$ (respectively $b = 0$), the stream function (2.3.7) depends only on the y coordinate (respectively x coordinate) and the flow becomes essentially one-dimensional. In each of these cases, $f \equiv 0$ and both the Lychagin–Rubtsov metric and its pullback are everywhere degenerate and their curvatures are ill-defined.

2.4 Chapter Summary

In this chapter, we have summarised key definitions from the framework of Monge–Ampère geometry. Using said framework, we have demonstrated that the covariant Navier–Stokes equations for homogeneous, incompressible fluid flows in two dimensions take the form of a genuine Monge–Ampère equation and the associated Monge–Ampère structure (2.2.2) was provided. Solutions to the Navier–Stokes equations were then shown to correspond to bi-Lagrangian submanifolds $\iota : L \hookrightarrow T^*M$ of the cotangent bundle, a subclass of which are described by local sections. Utilising results of [15], an almost (para-)complex structure (2.2.3) and almost (para-)Hermitian form (2.2.4) were associated with the Monge–Ampère structure and used to induce the Lychagin–Rubtsov metric (2.2.5) on the cotangent bundle. The dominance of vorticity and strain was shown to be determined by the signs of the Lychagin–Rubtsov metric and its pullback (2.2.7) to a Lagrangian submanifold described by a section, and in turn by the sign of the modified Laplacian of pressure $\mathring{\Delta}_{\mathrm{B}} p + \mathring{R}_{ij}(\mathring{\nabla}^i \psi)(\partial_i \psi)$, generalising the Weiss criterion to flows with curved background. It was then shown that the curvature (2.2.13), of the pullback metric is given by gradients of vorticity and strain and can be used to obtain topological information about the flow, for example via the Gauß–Bonnet theorem. We have seen an example of how these criterion manifest in the Euclidean plane, in the form of the Taylor–Green vortex. In the next chapter, we shall introduce the language of k-plectic geometry and reformulate the problem in terms of a Monge–Ampère structure with T-dual symplectic form, in order to generalise to higher dimensional flows.

3

k-Plectic Geometry and Incompressible Fluid Flows

While symplectic geometry proves to be a suitable framework for considering incompressible flows in two dimensions, note that incompressibility is not a natural consequence of the geometry. Rather, we assume that the flow is incompressible, such that (locally) a stream function determining the velocity exists, then solve the Poisson Monge–Ampère equation for that stream function. It turns out that the language of k-plectic geometry not only provides a way of encoding the incompressibility via a differential form, but also allows us to generalise our work so far to higher dimensional fluid flows, where the flow is not described by a stream function (rather, a stream form (1.2.8)) and hence does not have a genuine Monge–Ampère equation.

3.1 k-Plectic Geometry

In this section we provide a summary of the key facts from k-plectic geometry that are required in subsequent computations. For further details of the field, we refer the interested reader to works [23] and [24, 25]. To begin:

Definition 3.1.1 (k-Plectic Vector Space)
Let V be a real vector space. A differential $(k+1)$-form $\varpi \in \bigwedge^{k+1} V^*$ is called non-degenerate precisely when the contraction map $\lrcorner : V \to \bigwedge^{k+1} V^*$ given by $v \to v \lrcorner \varpi$, is injective on V. We then call (V, ϖ) a k-plectic vector space.

Note that, in general, the contraction map is not surjective. However, consider when $k = 1$ and ϖ is a non-degenerate 2-form. In particular, ϖ is a symplectic form on V and we recover the standard case of a symplectic vector space. Further, injectivity of the contraction map implies surjectivity by the rank–nullity theorem and hence $V \cong V^*$.

In order to define a k-plectic analogue of a Lagrangian subspace, we require a suitable generalisation of the orthogonal complement of a vector subspace $U \subseteq V$. Note that as a k-form takes k-vectors, there are k different classes of orthogonal complement, which we index as follows:

Definition 3.1.2 (ℓ-th Orthogonal Complement)
For $U \subseteq V$ a vector subspace of k-plectic vector space (V, ϖ), we define the ℓ-th orthogonal complement $U^{\perp,\ell}$ for $\ell = 1, \ldots, k$ with respect to ϖ by

$$U^{\perp,\ell} \ := \ \{v \in V \,|\, v \lrcorner u_1 \ldots \lrcorner u_\ell \lrcorner \varpi = 0 \textit{ for all } u_1, \ldots u_\ell \in U\} \ . \tag{3.1.1}$$

The vector subspace U is called an ℓ-Lagrangian subspace of V if and only if $U = U^{\perp,\ell}$ for some $\ell = 1, \ldots, k$. For $k = 1$, we can only have $\ell = 1$, which corresponds to the usual notion of a Lagrangian subspace with respect to the symplectic form ϖ. Note, unlike Lagrangian subspaces, which all have dimension $\frac{1}{2}\dim(V)$, for $k > 1$, ℓ-Lagrangian subspaces may have different dimensions. We conclude this section by extending the above definitions to manifolds:

Definition 3.1.3 (k-Plectic Manifolds)
Let $\varpi \in \Omega^{k+1}(N)$ be a point-wise non-degenerate differental $(k+1)$-form on an (almost k-plectic) manifold N. If, in addition, ϖ is closed, then N is a k-plectic manifold with k-plectic structure form ϖ.

It is a standard exercise to show that, when $k = 1$ and N is $2m$-dimensional, non-degeneracy of a differential 2-form ϖ is equivalent to $\varpi \wedge \ldots \wedge \varpi \neq 0 \in \Omega^{2m}(N)$, which is precisely the sense we applied in earlier sections.

Definition 3.1.4 (ℓ-Lagrangian Submanifolds)
Let (N, ϖ) be a k-plectic manifold, with submanifold $\iota : L \hookrightarrow N$. For $\ell = 1, \ldots, k$, let $(T_pL)^{\perp,\ell}$ denote the ℓ-th orthogonal complement of the tangent space to L at $p \in L$. We define the corresponding bundle

$$TL^{\perp,\ell} \ := \ \bigcup_{p \in L} \left\{ (p, X_p) \,\middle|\, X_p \in (T_pL)^{\perp,\ell} \right\} . \tag{3.1.2}$$

We call L an ℓ-Lagrangian submanifold of N if and only if $TL = TL^{\perp,\ell}$ for some $\ell = 1, \ldots, k$.

3.2 The Bridge Between Dimensions

Before discussing fluid flows in higher dimensions, let us revisit the two-dimensional problem and provide an alternative formulation. In Section 2.2, we have seen that the Monge–Ampère structure (2.2.2) encodes incompressible fluids on a two-dimensional Riemannian manifold $(M, \mathring{g})$. As before, let (x^i, q_i) be local coordinates on T^*M. This time, rather than using the standard symplectic structure (2.2.2) on T^*M, we now propose to take the 'T-dual' form

$$\varpi \ := \ \mathring{\nabla} q_i \wedge \star_{\mathring{g}} \mathrm{d}x^i \ , \tag{3.2.1}$$

that is, (2.2.4) without the pre-factor. It should be evident that ϖ is non-degenerate[1] and shall demonstrate in Section 3.3 that it is closed, hence is a symplectic (read: 1-plectic) form in two dimensions. It is then easily seen that the condition $\iota^*\varpi = 0$ with $\iota : L \hookrightarrow T^*M$ given by

$$\iota : x^i \mapsto (x^i, q_i) := (x^i, v_i(x)) , \tag{3.2.2}$$

where $v_i = v_i(x)$ are the components of the velocity (co-)vector field in local coordinates, is equivalent to requiring the incompressibility condition (1.2.3b). Thus, a Lagrangian submanifold L of T^*M is again obtained, this time with respect to ϖ rather than ω, such that L encodes incompressibility.

Moreover, using that in two dimensions, $\star_{\mathring{g}}(\mathrm{d}x^i \wedge \mathrm{d}x^j) = \sqrt{\det(\mathring{g})}\varepsilon^{ij}$ and the volume form on M is given by $\mathrm{vol}_M := \frac{\sqrt{\det(\mathring{g})}}{2}\varepsilon_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$, we may rewrite the Monge–Ampère form α defined in (2.2.2) as

$$\alpha = \tfrac{1}{2}\mathring{\nabla} q_i \wedge \mathring{\nabla} q_j \wedge \star_{\mathring{g}}(\mathrm{d}x^i \wedge \mathrm{d}x^j) - \hat{f}\,\mathrm{vol}_M . \tag{3.2.3}$$

As for ϖ, we will demonstrate in the next section that α is closed in arbitrary dimension and, as discussed around (2.2.2), α is non-degenerate in two dimensions precisely when $\hat{f} \neq 0$. Additionally, the requirement $\iota^*\alpha = 0$ under (3.2.2) is equivalent to the pressure constraint (1.2.3c), provided that we simultaneously demand that $\iota^*\varpi = 0$. Further, as discussed around equation (1.2.8), for incompressible flows, it is possible to find a stream function $\psi \in \mathscr{C}^\infty(U)$ on an open, contractible set $U \subseteq M$, such that $v = \star_{\mathring{g}}\mathrm{d}\psi$. Upon describing the velocity in this manner, we recover the Monge–Ampère equation presentation of the pressure constraint (2.2.1) on U. That is, the submanifolds given by pulling back α and ϖ via $\star_{\mathring{g}}\mathrm{d}\psi$, are precisely those given by the pullback of α and ω via $\mathrm{d}\psi$, as discussed in Chapter 2, hence correspond to smooth classical solutions our Monge–Ampère equation on U. Notice that we also have $\alpha \wedge \varpi = 0$ so that the pair (ϖ, α) is again a Monge–Ampère structure.

We may now follow our discussion in Section 2.2 and define an endomorphism $\hat{\mathcal{J}}$ of the tangent bundle $\mathfrak{X}(T^*M)$ by

$$\frac{\alpha}{\sqrt{|\hat{f}|}} =: \hat{\mathcal{J}} \lrcorner \varpi , \tag{3.2.4}$$

under the assumption that $\hat{f}$ does not vanish. As before, $\hat{\mathcal{J}}^2 = -\mathrm{sgn}(\hat{f})$, hence $\hat{\mathcal{J}}$ is an almost complex structure when $\hat{f} > 0$ and an almost para-complex structure when $\hat{f} < 0$. We make the following observation concerning $\hat{J}$ in two dimensions, which will be pertinent when we consider three dimensional flows in the following two chapters: Letting ε denote the dual poly-vector field to the Liouville volume form $\frac{1}{2}\omega^2$, it is possible to rewrite (3.2.4) as

$$\hat{\mathcal{J}}X = \frac{1}{\sqrt{|\hat{f}|}}\varepsilon \lrcorner (\varpi \wedge X \lrcorner \alpha) \quad \text{for all} \quad X \in \mathfrak{X}(T^*M) . \tag{3.2.5}$$

[1] In-fact, we have already seen this in our discussion around (2.2.4).

As in Section 2.2, we can always find a differential two-form $\hat{\mathcal{K}}$ of type $(1,1)$ with respect to $\hat{\mathcal{J}}$ such that $\hat{\mathcal{K}} \wedge \varpi = 0$, $\hat{\mathcal{K}} \wedge (\hat{\mathcal{J}} \lrcorner \varpi) = 0$, and $\hat{\mathcal{K}} \wedge \hat{\mathcal{K}} \neq 0$. In particular, we choose

$$\hat{\mathcal{K}} \ := \ \mathrm{sgn}(\hat{f})\sqrt{|\hat{f}|}\,\mathring{\nabla} q_i \wedge \mathrm{d}x^i \ , \tag{3.2.6}$$

that is, the standard symplectic structure (2.2.2) with the negative of the scale factor that ϖ has in (2.2.4). Importantly, the compatibility of $\hat{\mathcal{K}}$ and $\hat{\mathcal{J}}$, given by $\mathcal{K}(X,Y) = \mathrm{sgn}(\hat{f})\mathcal{K}(\mathcal{J}X, \mathcal{J}Y)$, again yields the metric (2.2.5). Note that the Lychagin–Rubtsov theorem again yields that the endomorphism (3.2.4) is integrable if and only if $\hat{f}$ is constant. This coincides precisely with when (3.2.6) is closed and is hence promoted from an almost (para-)Hermitian form to a Kähler form. In turn, $\hat{f} \neq 0$ is a necessary and sufficient condition for (2.2.5) to be a Kähler metric, this time with respect to $\hat{\mathcal{K}}$ and $\hat{\mathcal{J}}$. We shall show in Section B.2 that this can be refined to a hyper-Kähler structure à la [7].

In conclusion, the Monge–Ampère structure (ϖ, α), with ϖ defined by (3.2.1) and α written as (3.2.3), represents an alternative means to describe two-dimensional incompressible fluids. Whilst the Monge–Ampère structure (2.2.2) yields manifestly the description of the fluid flow in terms of a stream function and a genuine Monge–Ampère equation, the advantage of this alternative Monge–Ampère structure is that with this choice, we can straightforwardly generalise our treatment to fluid flows in any dimension, as we shall explain shortly.

Remark 3.2.1 (Choices of Differential Form)
At this stage, it is worth noting how our above choices deviate from constructions used in previous works. It is clear that (3.2.1) *and* (3.2.3) *are precisely a covariantisation of the Monge–Ampère structure in [11, 8], with* (3.2.4) *the corresponding almost (para-)complex structure. However, we are free to make a choice of differential two-form above, which corresponds to a choice of almost (para)-Hermitian metric on T^*M. In particular, [11] fix the non-degenerate bilinear form*

$$\hat{g}_\alpha(X,Y) \ := \ \frac{[(X \lrcorner \alpha) \wedge (Y \lrcorner \varpi) + (Y \lrcorner \alpha) \wedge (X \lrcorner \varpi)] \wedge \mathrm{vol}_M}{\frac{1}{2}\varpi^2} \tag{3.2.7}$$

for all $X, Y \in \mathfrak{X}(T^*M)$, as opposed to (2.2.5)*. As discussed in [5, 26], the third differential two-form may be defined by*[1] $\sqrt{|\hat{f}|}\hat{g}_\alpha(\hat{\mathcal{J}}-,-)$*, as opposed to* (3.2.6)*. Note also that the pullback of $\hat{g}_\alpha$ via* (3.2.2) *is then simply the Hessian of ψ without vorticity as a conformal factor, in contrast to* (2.2.7)*, where the vorticity is made manifest.*

3.3 Geometric Properties of Higher Dimensional Incompressible Fluid Flows

Having introduced the notion of k-plectic manifolds, we can now make precise the description of higher-dimensional incompressible fluid flows. The following formulation should make it trans-

[1] Note that whilst we present these expressions in our notation, the literature only treats the Euclidean case.

parent that our approach works in any dimension $m > 1$, with appropriate choices of volume form vol_M and sums over indices $i = 1, \ldots, m$. Our results shall be presented in a (predominantly) dimension-free manner though, where needed, we may specify to the case $m = 3$ for clarity.

Let M be a smooth m-dimensional Riemannian manifold. We consider again the pair of differential m-forms given by

$$\begin{aligned} \varpi &:= \mathring{\nabla} q_i \wedge \star_{\mathring{g}} \mathrm{d}x^i \ , \\ \alpha &:= \tfrac{1}{2} \mathring{\nabla} q_i \wedge \mathring{\nabla} q_j \wedge \star_{\mathring{g}} (\mathrm{d}x^i \wedge \mathrm{d}x^j) - \underbrace{\tfrac{1}{2}\big(\mathring{\Delta}_{\mathrm{B}} p + \mathring{R}^{ij} q_i q_j\big)}_{=: \hat{f}} \mathrm{vol}_M \end{aligned} \tag{3.3.1}$$

on T^*M where now $\mathrm{vol}_M := \frac{\sqrt{\det(\mathring{g})}}{m!} \varepsilon_{i_1,\ldots i_m} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_m}$. Note that these choices can be understood as a covariantisation of results previously presented in [8, 11] in three dimensions. Again, it can be verified that ϖ is non-degenerate and noting that

$$\mathrm{d}(\mathring{\nabla} q_i) = \tfrac{1}{2} \mathrm{d}x^l \wedge \mathrm{d}x^k \mathring{R}_{kli}{}^j q_j + \mathrm{d}x^j \mathring{\Gamma}_{ji}{}^k \wedge \mathring{\nabla} q_k \tag{3.3.2a}$$

and

$$\mathrm{d}\star_{\mathring{g}} \mathrm{d}x^i = -\mathring{g}^{jk} \mathring{\Gamma}_{jk}{}^i \, \mathrm{vol}_M \tag{3.3.2b}$$

it is straightforward to verify that ϖ is closed. It then follows that ϖ defines an $(m-1)$-plectic structure on T^*M. Additionally, the submanifold $\iota : L \hookrightarrow T^*M$ defined by $\iota^*\varpi = 0$ with ι given by (3.2.2) with $i = 1, 2, \cdots m$ is an m-dimensional $(m-1)$-Lagrangian submanifold of the $(m-1)$-plectic manifold (T^*M, ϖ). Furthermore, it follows from (3.3.2a) and

$$\mathrm{d}\star_{\mathring{g}} (\mathrm{d}x^i \wedge \mathrm{d}x^j) = 2\mathring{g}^{kl} \mathring{\Gamma}_{kl}{}^{[i} \star_{\mathring{g}} \mathrm{d}x^{j]} + 2\mathring{g}^{k[i} \mathring{\Gamma}_{kl}{}^{j]} \star_{\mathring{g}} \mathrm{d}x^l \tag{3.3.3a}$$

and

$$\mathrm{d}x^k \wedge \star_{\mathring{g}} (\mathrm{d}x^i \wedge \mathrm{d}x^j) = -2\mathring{g}^{k[i} \star_{\mathring{g}} \mathrm{d}x^{j]} \tag{3.3.3b}$$

that α is closed. It is also non-degenerate and so, the pair (T^*M, α) defines an $(m-1)$-plectic manifold. However, in general, while α is $(m-1)$-plectic, its pullback via (3.2.2) may not define an m-dimensional $(m-1)$-Lagrangian submanifold. As discussed above, the conditions $\iota^*\varpi = 0$ and $\iota^*\alpha = 0$ are equivalent to the incompressibility condition (1.2.3b) and the pressure constraint (1.2.3c), respectively. Note also that $\varpi \wedge \omega = 0$ and $\alpha \wedge \omega = 0$ for $\omega = \mathring{\nabla} q_i \wedge \mathrm{d}x^i$ the standard symplectic structure on T^*M and so, ϖ and α are Monge–Ampère forms for ω. In contrast, the wedge product $\alpha \wedge \varpi \equiv 0$ if and only if $m \neq 3$.

Next, we wish to treat the relation (3.2.4) in higher dimensions. To this end, we shall specialise to the case $m = 3$ and use the results of [9]. In particular, we note that the there is an isomorphism $\phi : \Omega^5(T^*M) \to \mathfrak{X}(T^*M) \otimes \Omega^6(T^*M)$ that is induced by the natural exterior product pairing $\Omega^1(T^*M)\otimes\Omega^5(T^*M)\to\Omega^6(T^*M)$.[1] Consequently, given a differential 3-form α and a volume form vol on T^*M, it is then possible to define the so-called Hitchin endomorphism $\hat{\mathcal{J}}^\alpha_{\rm vol}:\mathfrak{X}(T^*M)\to\mathfrak{X}(T^*M)$ by

$$\hat{\mathcal{J}}^\alpha_{\rm vol}(X)\,{\rm vol} \;:=\; \phi(\alpha\wedge X\lrcorner\,\alpha)\,. \tag{3.3.4}$$

Now let vol $=\frac{1}{3!}\omega^3$, the Liouville volume form with respect to the standard symplectic form ω on T^*M and let ε denote the poly-vector field dual to the Liouville volume form, that is, $\varepsilon\lrcorner\frac{1}{3!}\omega^3=1$. Using (3.3.4), we may then associate with the differential three-form α defined in (3.3.1) the endomorphism

$$\hat{\mathcal{J}}X \;:=\; -\frac{1}{2\sqrt{|\hat f|}}\,\varepsilon\lrcorner\,(\alpha\wedge X\lrcorner\,\alpha)\quad\text{for all}\quad X\;\in\;\mathfrak{X}(T^*M) \tag{3.3.5}$$

under the assumption that $\hat f$ does not vanish. It then follows that $\hat{\mathcal{J}}$ is an almost complex structure on T^*M when $\hat f>0$ and an almost para-complex structure when $\hat f<0$. Furthermore, the differential two-form $\hat{\mathcal{K}}$ defined in (3.2.6), with now i running from one to three, together with (3.3.5) satisfy $\hat{\mathcal{K}}(\hat{\mathcal{J}}X,Y)=-\hat{\mathcal{K}}(X,\hat{\mathcal{J}}Y)$ for all $X,Y\in\mathfrak{X}(T^*M)$, hence $\hat{\mathcal{K}}$ is an almost (para-)Hermitian form. Consequently, we can define an almost (para-)Hermitian metric $\hat g$ on T^*M with respect to (3.3.5) by setting $\hat g(X,Y):=\hat{\mathcal{K}}(X,\hat{\mathcal{J}}Y)$ for all $X,Y\in\mathfrak{X}(T^*M)$. Explicitly,

$$\hat g \;=\; \tfrac12\hat f\mathring{g}_{ij}{\rm d}x^i\odot{\rm d}x^j+\tfrac12\mathring{g}^{ij}\mathring{\nabla}q_i\odot\mathring{\nabla}q_j\;. \tag{3.3.6}$$

In contrast to Remark 3.2.1, in three dimensions, this metric is essentially a covariantisation of a bilinear form introduced in [15] (see also [8]), from which it follows that $\hat{\mathcal{K}}$ is of type $(1,1)$ with respect to (3.3.5).

Noting that (3.3.6) is a direct generalisation of (2.2.5) from two to three dimensions, we return to our arbitrary dimensional formulation and call metrics of this form, with $i,j=1,\ldots,m$, a Lychagin–Rubtsov metric on T^*M. In view of our later applications, we provide an expression for the curvature scalar of the metric (3.3.6). The following is derived in Section C.2.3 and holds in any dimension:

$$\begin{aligned}\hat R \;=\;& \frac{1}{\hat f}\mathring{R}-\frac{1}{4\hat f^2}\mathring{R}_{ijk}{}^l\mathring{R}^{ijkm}q_kq_m-(m-1)\hat\Delta_{\rm B}\log\big(|\hat f|\big)-\mathring{g}_{ij}\frac{\partial^2}{\partial q_i\partial q_j}\log\big(|\hat f|\big)\\ &+\frac{1}{4\hat f}(m-1)(m-2)\mathring{g}^{ij}\left(\frac{\partial}{\partial x^i}+\mathring{\Gamma}_{ik}{}^lq_l\frac{\partial}{\partial q_k}\right)\log\big(|\hat f|\big)\left(\frac{\partial}{\partial x^j}+\mathring{\Gamma}_{jm}{}^nq_n\frac{\partial}{\partial q_m}\right)\log\big(|\hat f|\big)\\ &\quad+\frac14 m(m-3)\mathring{g}_{ij}\frac{\partial}{\partial q_i}\log\big(|\hat f|\big)\frac{\partial}{\partial q_j}\log\big(|\hat f|\big)\;,\end{aligned} \tag{3.3.7}$$

[1] Explicitly, $\phi:\Omega^5(T^*M)\to\mathfrak{X}(T^*M)\otimes\Omega^6(T^*M)$ is given by $\phi(\rho)(\lambda,X_1,\ldots,X_6):=X_1\lrcorner\ldots\lrcorner X_6\lrcorner(\rho\wedge\lambda)$ for all $\rho\in\Omega^5(T^*M)$, $\lambda\in\Omega^1(T^*M)$, and $X_1,\ldots,X_6\in\mathfrak{X}(T^*M)$.

where $\hat{\Delta}_{\mathrm{B}}$ is the Beltrami Laplacian for $\hat{g}$.

Finally, let us present a formula for the pullback of the metric (3.3.6) in arbitrary dimension, which utilises the notation of fluid dynamics. Note first that pulling back $\hat{f}$ via (3.2.2) and applying the pressure equation (1.2.3c) (in the form (1.2.7)) yields

$$f \;:=\; \iota^*\hat{f} \;=\; -\tfrac{1}{2}A_j{}^i A_i{}^j\,, \tag{3.3.8}$$

where $A_i{}^j = A_{ik}\mathring{g}^{kj}$ and A_{ij} is the velocity gradient tensor defined in (1.2.5). Additionally, observe that the pullback of $\mathring{\nabla} q_i$ via (3.2.2) is given by

$$\iota^*(\mathring{\nabla} q_i) \;=\; \mathring{\nabla}_j v_i \mathrm{d}x^j \;=\; A_{ij}\mathrm{d}x^j\,. \tag{3.3.9}$$

It then follows that the pullback of (3.3.6) is given by

$$g \;=\; \tfrac{1}{2}g_{ij}\mathrm{d}x^i \odot \mathrm{d}x^j \quad \text{with} \quad g_{ij} \;:=\; \mathring{g}^{kl}A_{ki}A_{lj} - \tfrac{1}{2}A_k{}^l A_l{}^k \mathring{g}_{ij}\,. \tag{3.3.10}$$

We can demonstrate that for incompressible two-dimensional fluid flows, this is precisely equivalent to the pullback of (2.2.5) via a section $\mathrm{d}\psi$ on open, contractible neighbourhoods. Namely, (2.2.7) may be recovered from (3.3.10) by noting that $A_{ij} = \mathring{\nabla}_j v_i$ and that for two-dimensional incompressible flows on an open contractible neighbourhood $U \subseteq M$, the statement (1.2.8) is simply

$$v^i \;=\; \sqrt{\det(\mathring{g})}\epsilon^{ij}\partial_j\psi\,. \tag{3.3.11}$$

However, note that while in two dimensions we have directly related the signature of (2.2.7) to the sign of (3.3.8), there is no such apparently simple relation in higher dimensions; in Section 3.4 we will present an explicit example demonstrating that the metric (3.3.10) does not even have to be singular along the curves given by $f = 0$. While it is not yet fully understood why this is the case, in the next chapter we shall demonstrate how this is also true of compressible flows in two dimensions given by symmetry reduction.

3.3.1 Topology of Three-Dimensional Incompressible Fluid Flows

In two dimensions, we utilised the local Gauß–Bonnet theorem (2.2.14) in order to relate the geometry of fluid flows, as described by the curvature scalar (2.2.13), to a topological invariant, namely the Euler characteristic of a given compact region. In three dimensions (in-fact in any odd number of dimensions), it quickly becomes apparent that this is not a suitable approach and that we require alternative topological quantity.

Recall now the standard symplectic form ω and its associated tautological one-form $\theta := q_i\mathrm{d}x^i$ satisfying $\omega = \mathrm{d}\theta$. Their respective pullbacks under (3.2.2) are $\iota^*\omega = \mathrm{d}v = \tfrac{1}{2}\zeta_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$ and $\iota^*\theta = v = v_i\mathrm{d}x^i$ and it should be noted that $\iota^*\omega$ vanishes if and only if the vorticity two form (1.2.4) is zero. It then follows from the distributive property of pullbacks over the wedge product that

$$\iota^*(\theta \wedge \mathrm{d}\theta) \;=\; v_i \zeta^i \,\mathrm{vol}_{M^3} \quad \text{with} \quad \zeta^i \;:=\; \tfrac{1}{2}\sqrt{\det(\mathring{g}_3)}\,\varepsilon^{ijk}\zeta_{jk} \tag{3.3.12}$$

is the vorticity in three dimensions, derived from (1.2.4). Integrals of quantities of this form, over a compact region $U \subseteq M^3$, are referred to as helicity [27, 28]. Hence, in our context $v_i\zeta^i$ may be referred to as the helicity per volume.

Consider an inviscid, incompressible fluid, with kinematics described by the Euler equations, on a compact region $U \subseteq M^3$. Suppose also that U describes the volume contained inside some closed orientable surface, which is moving with the fluid and has (continuous, outward) unit normal n with components denoted n_i. It is shown[1] in [27] that, provided the distribution of vorticity is local and continuous, with $n_i\zeta^i = 0$, then the integral of (3.3.12) is an invariant of the Euler equations and the vorticity field within the volume is conserved. Furthermore, it is shown in [29, 30] that for discrete vortex filaments, this quantity can be associated with the topological invariants given by the Gauß linking number and Călugăreanu invariant of [31, 32]. Whitehead [33] also showed that helicities are isotopy invariants of their volume. Perhaps more significantly, a recent work [34] has managed to demonstrate that, in ideal conditions, helicity-type quantities can be reinterpreted as Abelian Chern–Simons actions and hence can be related to the Jones polynomial.

Observe that, in addition to the interpretation of the pullbacks of (3.3.1) under (3.2.2) as the incompressibility and pressure equations, in three dimensions we now have that the corresponding pullback of the standard symplectic form encodes the helicity. Additionally, previous work relating helicity to various topological invariants suggests that, as in two dimensions, we can relate the topology of fluid flows to our geometric constructions.

3.4 Incompressible Fluids in Euclidean Space - Examples

For completeness, we now provide an example computation for a three-dimensional incompressible fluid flow. To demonstrate the usefulness of our covariant formulation, we consider a flow on flat background in cylindrical polar coordinates. Note that we present the curvatures of the Lychagin–Rubtsov metric and its pullback, despite these having no natural interpretation via the Gauß–Bonnet theorem in three dimensions. However, we shall wish to compare these curvatures to those we shall obtain upon our return to three-dimensional examples at the end of Chapter 4, where we shall address how coordinate symmetries of the underlying manifold can be used to produce a (potentially compressible) two-dimensional flow, with its own Lychagin–Rubtsov metric and alternative diagnostics for vorticity and strain.

[1] See also [28] for discussion in the context of magneto-hydrodynamics.

3.4.1 Hill's Vortex

To set the stage, we consider flows on $M = \mathbb{R}^2 \times_{r^2} S^1$ equipped with the metric

$$\mathring{g} \;=\; \mathrm{d}r \otimes \mathrm{d}r + \mathrm{d}z \otimes \mathrm{d}z + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta\,, \tag{3.4.1}$$

where $r \in \mathbb{R}^+$, $z \in \mathbb{R}$, and $\theta \in [0, 2\pi)$, inducing standard cylindrical coordinates. The non-vanishing Christoffel symbols of $\mathring{g}$ are given by

$$\mathring{\Gamma}_{\theta\theta}{}^{r} \;=\; -r \quad \text{and} \quad \mathring{\Gamma}_{r\theta}{}^{\theta} \;=\; \mathring{\Gamma}_{\theta r}{}^{\theta} \;=\; \tfrac{1}{r}\,, \tag{3.4.2}$$

from which it follows that (3.4.1) is flat and $\mathring{R} = 0$. Therefore, we have that $\hat{f} = f = \frac{1}{2}\mathring{\Delta}_{\mathrm{B}}p$ for flows on M. Pulling back (3.3.1) by (3.2.2) yields the equations (1.2.3b) and (1.2.3c), which simplify to

$$\partial_r v_r + \partial_z v_z + \tfrac{1}{r^2}\partial_\theta v_\theta + r v_r \;=\; 0\;, \tag{3.4.3a}$$

and

$$\tfrac{1}{2}\mathring{\Delta}_{\mathrm{B}}p \;=\; (\partial_r v_r)(\partial_z v_z) - (\partial_r v_z)(\partial_z v_r) - (\tfrac{v_r}{r})^2 + \tfrac{1}{r^4}\left[v_\theta^2 + \tfrac{r}{2}\partial_r(v_\theta^2) + r v_\theta(\partial_\theta v_r) - (\partial_\theta v_\theta)^2\right]\;, \tag{3.4.3b}$$

respectively, in our coordinates. We may now fix components of velocity satisfying the above pair of equations, in order to study specific flow regimes well suited to cylindrical coordinates.

We wish to showcase vortices of Hicks-Moffatt type [35, 27] - isolated, spherical regions of local and continuous vorticity, placed in a uniform flow, parallel with the z axis. For an in-depth review of such vortices, we direct the interested reader to [36]. We shall normalise the speed of the uniform flow and the radius of the sphere to 1 for simplicity. Fix the notation $\sigma(r, z) = \sqrt{r^2 + z^2}$ and call points with $\sigma^2 \lessgtr 1$ the interior/exterior. The components of velocity for Hicks-Moffatt vortices are then written as follows

$$v_r \;=\; -\tfrac{1}{r}\partial_z\psi\,, \quad v_z \;=\; \tfrac{1}{r}\partial_r\psi\,, \quad \text{and} \quad v_{\theta,\kappa} \;=\; \tfrac{1}{r}\kappa\psi\,, \tag{3.4.4}$$

where $\psi = \psi(r, z)$ and κ parametrises the swirl of the flow. Note that (3.4.3a) is trivially satisfied by such a choice. It is important to observe that the helicity (3.3.12), is non-zero if and only if the flow has non-zero swirl [37, 38], hence it is possible to have regions of vorticity in three dimensions, which have vanishing helicity. Further, the function ψ (which we shall refer to as the stream function for reasons revealed in Section 4.4.2) is prescribed on the exterior and interior in turn, such that the piecewise function is continuous (in-fact, vanishing) on the boundary of the sphere given by $\sigma^2 = 1$ and (3.4.3b) is satisfied.

In particular, on the exterior of the sphere, we choose

$$\psi_{\mathrm{ext}}(r, z) \;:=\; \tfrac{1}{2}r^2\left(1 - \tfrac{1}{\sigma^3}\right)\,. \tag{3.4.5a}$$

such that the flow far from the sphere is uniform with unit speed directed along the z axis, with some rotation in the θ direction, as dictated by κ. On the interior, we set

$$\psi_{\text{int},\kappa}(r,z) \;:=\; \tfrac{3}{2}r^2\left(b(\kappa) - c(\kappa)\frac{J_{\frac{3}{2}}(\kappa\sigma)}{(\kappa\sigma)^{\frac{3}{2}}}\right), \tag{3.4.5b}$$

with

$$b(\kappa) \;:=\; \frac{J_{\frac{3}{2}}(\kappa)}{\kappa J_{\frac{5}{2}}(\kappa)} \quad\text{and}\quad c(\kappa) \;:=\; \frac{\sqrt{\kappa}}{J_{\frac{5}{2}}(\kappa)}, \tag{3.4.5c}$$

where $J_n(x)$ is the n-th order Bessel function with argument x.[1] Studying such examples for arbitrary κ becomes cumbersome and somewhat unenlightening, so let us specify to the limiting case when $\kappa = 0$ and the flow has vanishing helicity; this choice corresponds to Hill's spherical vortex [40]. Dropping the $\kappa = 0$ subscripts, the θ component of velocity, (3.4.4) then becomes $v_\theta = 0$.

The exterior problem is largely trivial, so we shall only discuss it briefly here. Since the vorticity of our flow is isolated on the interior, it follows that strain dominates everywhere on the exterior and, as should be expected from previous discussions, $\hat{f}$ is everywhere negative, $\hat{g}$ is everywhere Kleinian, with everywhere negative curvature scalar and no singularities on the domain. There is, however, a coordinate singularity in g, along the line $z = 0$, corresponding to where the exterior flow passes the equator of the sphere, but not to any sign change in f or ζ_{ij}. This may be due to the fact that $\zeta_{ij} = 0$ on the exterior, though more investigation would need to be undertaken.

Turning to the interior solution for Hill's vortex, the stream function (3.4.5b) simplifies to

$$\psi(r,z) \;:=\; \psi_{\text{int},0}(r,z) \;=\; \tfrac{3}{4}r^2\big(r^2 + z^2 - 1\big). \tag{3.4.6}$$

Applying (3.4.4) then yields the remaining velocity components

$$v_r \;=\; -\tfrac{3}{2}rz \quad\text{and}\quad v_z \;=\; \tfrac{3}{2}\big(2r^2 + z^2 - 1\big). \tag{3.4.7}$$

Imposing that (3.4.3b) is satisfied, we find that the Laplacian of pressure is given by

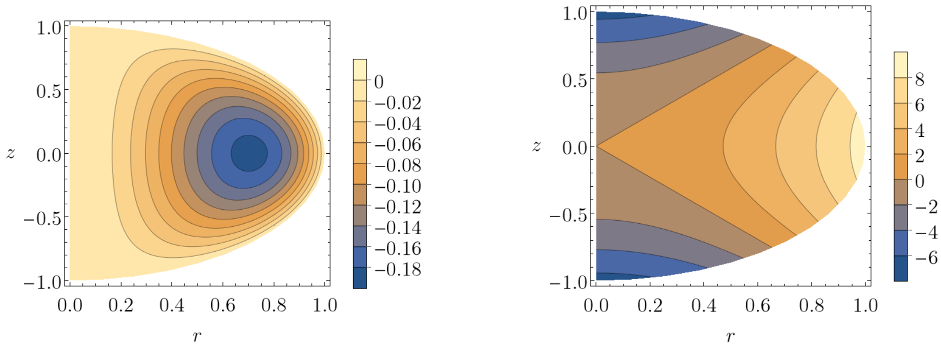
$$f \;=\; \tfrac{1}{2}\mathring{\Delta}_{\mathrm{B}}p \;=\; \tfrac{9}{4}\left(4r^2 - 3z^2\right) \tag{3.4.8}$$

and the curvature (3.3.7) of the Lychagin–Rubtsov metric (3.3.6) is given by

$$\hat{R} \;=\; \frac{8\left(8r^4 + 69r^2z^2 - 9z^4\right)}{9r^2\left(4r^2 - 3z^2\right)^3}. \tag{3.4.9}$$

It follows that $\hat{g}$ is Riemannian with positive scalar curvature where $4r^2 > 3z^2$ and Kleinian with curvature of indefinite sign when $4r^2 < 3z^2$. Further, along the contour $f = 0$, the metric $\hat{g}$ exhibits a curvature singularity, see Figure 3.4.3a.

[1] Such an explicit solution was found in the context of magneto-hydrodynamics by Prendergast [39]. In the context of Navier–Stokes , solutions of this type are also referred to as Hill's spherical vortex with swirl.



(a) Contours for $\psi_{\text{int},0}$ at constant θ. The contours are closed and concentric, forming toroidal vortex tubes when rotated around the z axis.

(b) The Laplacian of pressure function f, which vanishes along $4r^2 = 3z^2$, is positive between these curves and negative outside of them.

Figure 3.4.1: Contour plots of the stream function (3.4.6) (left) and Laplacian of pressure quantity (3.4.8) (right) for the interior of Hill's unit spherical vortex. Note that the region on which $f > 0$ contains closed streamlines of ψ with sufficiently large magnitude.

We now wish to evaluate the pullback of the Lychagin–Rubtsov metric via the velocity components, using the velocity gradient tensor (1.2.5) which, for Hill's vortex, is given by

$$A_{ij} = \begin{pmatrix} -\frac{3}{2}z & -\frac{3}{2}z & 0 \\ 6r & 3z & 0 \\ 0 & 0 & -\frac{3}{2}r^2 z \end{pmatrix}. \tag{3.4.10}$$

Plugging this into (3.3.10) yields that the pullback metric via (3.2.2) is given by

$$g_{ij} = \frac{9}{4}\begin{pmatrix} 20r^2 - 2z^2 & 9rz & 0 \\ 9rz & 5r^2 + z^2 & 0 \\ 0 & 0 & r^2\left(4r^2 - 2z^2\right) \end{pmatrix}. \tag{3.4.11}$$

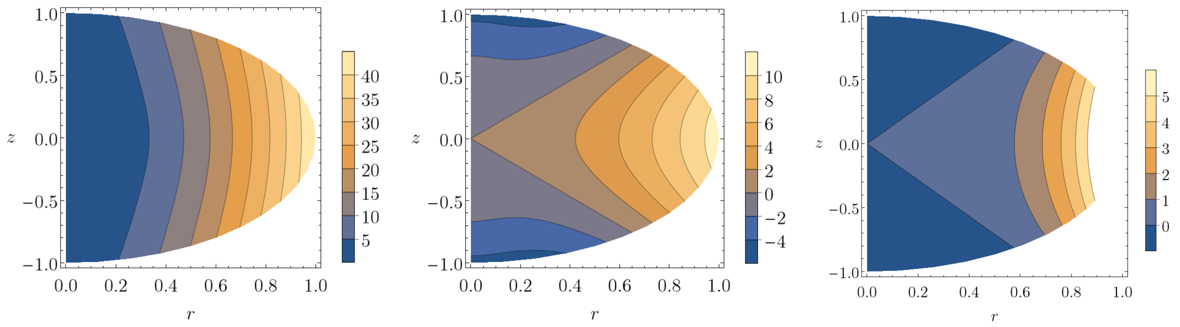
The eigenvalues of this metric are given by

$$E_{\pm} = \tfrac{9}{8}\left(25r^2 - z^2 \pm 3\sigma\sqrt{(25r^2 + z^2)}\right) \quad \text{and} \quad E_3 = \tfrac{9}{4}r^2\left(4r^2 - 2z^2\right) \tag{3.4.12}$$

and its curvature is given by

$$R = \frac{8\left(10000r^8 + 1880r^6z^2 - 2292r^4z^4 - 1229r^2z^6 - 85z^8\right)}{9\left(2r^2 - z^2\right)^2\left(100r^2 - 71r^2z^2 - 2z^4\right)^2}. \tag{3.4.13}$$

It follows that the metric (3.4.11) is Riemannian with positive scalar curvature when $2r^2 - z^2 > 0$ and $100r^4 - 71r^2z^2 - 2z^4 > 0$ simultaneously, and singular when there is equality. The



(a) Contours of E_+. This eigenvalue is positive on the whole domain and increases in magnitude as z increases.

(b) Contours of E_-, which vanishes along $100r^4 - 71r^2z^2 - 2z^4 = 0$, and is positive between the resulting lines.

(c) Contours of E_3. This eigenvalue vanishes along $4r^2 = 2z^2$, is positive between these lines, and negative outside of them.

Figure 3.4.2: Plots of the eigenvalues (3.4.12) of the pullback metric (3.4.11) for the interior solution of Hill's spherical vortex.

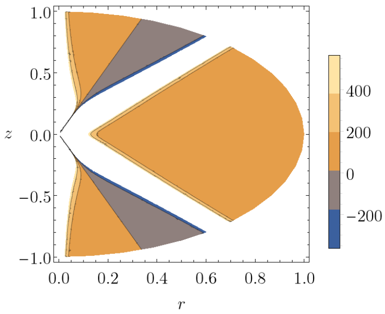
pullback metric is Kleinian elsewhere, with signature $(+1, -1, -1)$ and negative scalar curvature when $2r^2 - z^2 < 0$ and $100r^4 - 71r^2z^2 - 2z^4 < 0$ simultaneously. In the region where $2r^2 - z^2 > 0$ and $100r^4 - 71r^2z^2 - 2z^4 < 0$, the curvature has indeterminate sign and the metric is of signature $(+1, +1, -1)$. Observe that, unlike in previous problems, neither of the singularities occur precisely where $f = 0$, however the region where $f > 0$ is still within that where (3.4.11) is Riemannian. The only non-zero component of vorticity is given by

$$\zeta_{12} = \tfrac{15}{4} r \,, \tag{3.4.14}$$

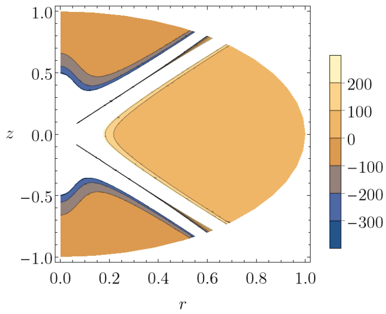
which vanishes along the z-axis, which lies within the Kleinian region, again in-line with the work of Larchevêque [4] and our observations in two dimensions.

3.5 Chapter Summary

In this chapter, we have introduced the language of k-plectic geometry and ℓ-Lagrangian submanifolds. It was noted that the Monge–Ampère-type structure given by (3.3.1) corresponds to the covariant, incompressible Navier–Stokes equations in arbitrary dimension and is equivalent to (2.2.2) in two dimensions. We again recovers an almost (para-)complex structure (3.2.4), almost (para-)Hermitian form (3.2.6), and Lychagin–Rubtsov metric (3.3.6). A formula (3.3.10) for the pullback of the Lychagin–Rubtsov metric via (3.2.2) in arbitrary dimension was provided, relating the geometry of Lagrangian submanifolds described by the velocity (co-)vector to the velocity gradient tensor (1.2.5). A potential approach for relating topological invariants in three



(a) Plot of the curvature $\hat{R}$ of the metric $\hat{g}$, which is singular along $r = 0$ and $4r^2 = 3z^2$, where $f = 0$.



(b) Plot of the curvature R of the metric g, which is singular along $4r^2 = 2z^2$ and $100r^4 - 71r^2z^2 - 2y^4 = 0$.

Figure 3.4.3: Contour plots of the curvatures (3.4.9) and (3.4.13) respectively.

dimensions to these geometric constructions was briefly discussed. We concluded by presenting an example of a flow in three-dimensional Euclidean space, hence demonstrating that, while signature change of the Lychagin–Rubtsov metric still coincides with sign change in $\hat{f}$ for higher dimensional flows, the signature of the pullback metric is no longer explicitly given by the sign of f, in contrast to the two-dimensional case. In the next chapter we consider a class of three-dimensional flows with symmetry and address the two-dimensional flow produced by reducing with respect to such a symmetry.

4 Dimensional Reductions

Earlier works [7, 11] considered elements of a class of solutions to the three-dimensional incompressible Euler and Navier–Stokes equations, with Euclidean background metric, which take the form [41]

$$(\dot{x}^1, \dot{x}^2, \dot{x}^3) \ := \ \big(v_1(x^1, x^2, t), v_2(x^1, x^2, t), x^3\gamma(x^1, x^2, t) + W(x^1, x^2, t)\big)\,, \tag{4.0.1}$$

for γ and W some functions, where the superposed dot refers to the derivative with respect to the time t. Such flows are referred to as 'two-and-a-half-dimensional' flows and are a generalisation of what is known as 'columnar flow' [42]. In particular, Burgers' vortex [43] is one such flow in $\mathbb{R}^3$ (for which $W \equiv 0$ and $\gamma = \gamma(t)$), treated in [11] in the following manner: Assume that $\mathbb{R}$ acts on the phase space $T^*\mathbb{R}^3$ via translation of x^3 and q_3, with infinitesimal generator described by the Killing vector field $\frac{\partial}{\partial x^3} + \gamma\frac{\partial}{\partial q_3}$. Given that such an action preserves the Monge–Ampère structure (3.3.1), for which the velocity components of Burgers' vortex define a solution in the sense of (3.2.2), there exists an equivalent two-dimensional Monge–Ampère equation and classical solution, replicating the so-called Lundgren transformation [44]. By performing this reduction via our geometric formulation, the resulting two-dimensional Monge–Ampère equation can then be studied in the sense of Chapter 2.

In this chapter, we shall consider solutions of the form (4.0.1) with $\gamma \equiv 0$ and $W = W(x^1, x^2, t)$ in more detail, as well as extending the treatment to flows on an arbitrary Riemannian manifold. That is, we consider the case when the underlying three-dimensional manifold M exhibits some symmetry in one particular coordinate x^3, along which we reduce. We will find that the three-dimensional, covariant, incompressible Navier–Stokes equations produce 'adapted' flow equations in two dimensions. While we drop the parameter time t for brevity, all functions and constants will be interpreted as being 'at fixed t' and hence are allowed to vary in time.

4.1 Setting for Reduction

We wish to consider flows on a three-dimensional background manifold M^3 of a specific form, such that reductions in one coordinate exist. In particular, let M^3 be a warped-product of a two-dimensional manifold M^2 with metric $\mathring{g}_2$, and a one-dimensional manifold N with local coordinates x^3. Then, there exists a metric on M^3 with the following form

$$\mathring{g}_3 \;=\; \mathring{g}_2 + \mathrm{e}^{2\varphi}\,\mathrm{d}x^3 \otimes \mathrm{d}x^3\,, \tag{4.1.1}$$

where we refer to $\varphi \in \mathscr{C}^\infty(M^2)$ as the warping factor. For the remainder of this chapter, we consider lower case indices $i, j, \ldots = 1, 2$ so, for example, $\mathring{g}_2 = \frac{1}{2}\mathring{g}_{ij}\mathrm{d}x^i \odot \mathrm{d}x^j$ in components. Observe that the only non-vanishing Christoffel symbols for $\mathring{g}_3$ are $\mathring{\Gamma}_{33}{}^i = \mathrm{e}^{2\varphi}\mathring{g}^{ij}\partial_j\varphi$, $\mathring{\Gamma}_{i3}{}^3 = \partial_i\varphi$ and the Christoffel symbols $\mathring{\Gamma}_{ij}{}^k$ for $\mathring{g}_2$.

Next, consider the differential forms (3.3.1) on M^m for $m = 2, 3$ and denote these by ϖ_m and α_m respectively. We shall use similar notation for other quantities occurring in both M^3 and M^2. Then, under the assumption that $p \in \mathscr{C}^\infty(M^2)$, some algebra reveals that

$$\begin{aligned}\varpi_3 \;&=\; \mathrm{e}^{\varphi}\,\varpi_2 \wedge \mathrm{d}x^3 + \mathrm{e}^{-\varphi}\,\mathrm{vol}_{M^2} \wedge \mathring{\nabla} q_3\,,\\ \alpha_3 \;&=\; \mathrm{e}^{\varphi}(\alpha_2 - \hat{h}_+\mathrm{vol}_{M^2}) \wedge \mathrm{d}x^3 + \mathrm{e}^{-\varphi}\,(\varpi_2 - q_3\mathrm{d}x^3 \wedge \star_{\mathring{g}_2}\mathrm{d}\varphi) \wedge \mathring{\nabla} q_3\,,\end{aligned} \tag{4.1.2a}$$

with

$$\hat{h}_\pm \;:=\; \tfrac{1}{2}\big[\mathring{\nabla}^i\varphi\partial_i p - \big(\mathring{\nabla}^i\mathring{\nabla}^j\varphi \pm \mathring{\nabla}^i\varphi\mathring{\nabla}^j\varphi\big)q_iq_j - \mathrm{e}^{-2\varphi}(\mathring{\Delta}_{\mathrm{B}}\varphi \pm \mathring{\nabla}^i\varphi\partial_i\varphi)q_3^2\big]\,, \tag{4.1.2b}$$

where all differential operators in $\hat{h}_\pm$ are with respect to the metric $\mathring{g}_2$. Unless explicitly labelled, this shall hold true for all formulae in the remainder of this chapter. Furthermore, we obtain

$$\begin{aligned}\varpi_2' \;&:=\; \partial_{x^3} \lrcorner\, \varpi_3\\ &=\; \mathrm{e}^{\varphi}\big(\varpi_2 + q_i\mathring{\nabla}^i\varphi\,\mathrm{vol}_{M^2}\big)\,,\\ \alpha_2' \;&:=\; \partial_{x^3} \lrcorner\, \alpha_3\\ &=\; \mathrm{e}^{\varphi}\big[\alpha_2 - \big(\hat{h}_+ + \mathrm{e}^{-2\varphi}\mathring{\nabla}^i\varphi\partial_i\varphi\, q_3^2\big)\,\mathrm{vol}_{M^2} + q_i\mathring{\nabla}^i\varphi\,\varpi_2\big] + \mathrm{e}^{-\varphi}q_3\mathrm{d}q_3 \wedge \star_{\mathring{g}_2}\mathrm{d}\varphi\,.\end{aligned} \tag{4.1.3}$$

A short calculation then shows that both ϖ_2' and α_2' are closed. In fact, using (3.3.2b), we also have that

$$\varpi_2' \;=\; \mathrm{d}(\star_{\mathring{g}_2}\mathrm{e}^{\varphi}q_i\mathrm{d}x^i)\,. \tag{4.1.4}$$

It follows also from (4.1.3) and Cartan's magic formula for Lie derivatives that

$$\mathcal{L}_{\partial_{x^3}}\varpi_3 = 0 = \mathcal{L}_{\partial_{x^3}}\alpha_3\,, \tag{4.1.5}$$

hence the actions of Lie groups with infinitesimal generator $\frac{\partial}{\partial x^3}$ preserve our Monge–Ampère structure (4.1.2). This suggests that, for flows on M^3 with metric (4.1.1) and the aforementioned assumptions, it may be possible to perform a dimensional reduction along the direction x_3. Crucially, if the Lie algebra of our symmetry is generated by $\frac{\partial}{\partial x^3}$ only (and is hence described by $\mathbb{R}$), the corresponding Lie group G must be one-dimensional and hence isomorphic to either $\mathbb{R}$ or S^1. In the subsequent two sections, we present two different approaches to this reduction, using symplectic and 2-plectic geometry in turn.

Remark 4.1.1 (Almost (Para-)Complex Forms and Dimensional Reductions)
Such reductions also enable us to make the relationship between (3.2.5) and (3.3.5) a bit more explicit. Let $\hat{\mathcal{J}}_m$ denote the almost (para-)complex form on $\mathfrak{X}(T^*M^m)$ and ω_m be the standard symplectic structure. In addition, we let ε_m be the poly-vector field dual to Liouville volume form on T^*M^m with respect to ω_m. As $\omega_2 \wedge \omega_2 = \varpi_2 \wedge \varpi_2$, ϵ_2 is also dual to the Liouville volume form of T^*M^2 with respect to ϖ_2. Also, let M^3 be a warped-product manifold with metric (4.1.1) as above and consider the special case when $\varphi = 0$. Then $M^3 = M^2 \times N$, with the metric

$$\mathring{g}_3 \;=\; \mathring{g}_2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 \ . \tag{4.1.6}$$

Assuming that $p \in \mathscr{C}^\infty(M^2)$, the formulae (4.1.2) simplify to

$$\varpi_3 \;=\; \varpi_2 \wedge \mathrm{d}x^3 + \mathrm{vol}_{M^2} \wedge \mathrm{d}q_3 \quad \text{and} \quad \alpha_3 \;=\; \alpha_2 \wedge \mathrm{d}x^3 + \varpi_2 \wedge \mathrm{d}q_3 \ . \tag{4.1.7}$$

The decomposition of α_3 and the effectiveness $\alpha_2 \wedge \varpi_2 = 0$ imply that $\alpha_3 \wedge (X \lrcorner \alpha_3) = -2(\varpi_2 \wedge X \lrcorner \alpha_2)\, \mathrm{d}q_3 \wedge \mathrm{d}x^3$ for all $X \in \mathfrak{X}(T^*M^2)$.[1] Since $\varepsilon_3 = \varepsilon_2 \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial q_3}$, this then yields $\varepsilon_3 \lrcorner (\alpha_3 \wedge X \lrcorner \alpha_3) = -2\varepsilon_2 \lrcorner (\varpi_2 \wedge X \lrcorner \alpha_2)$. Consequently, combining this result with (3.3.5) and (3.2.5), we finally obtain

$$\hat{\mathcal{J}}_3|_{M^2} \;=\; \hat{\mathcal{J}}_2 \ . \tag{4.1.8}$$

4.2 Symplectic Reduction

We begin this section by recalling the Marsden–Weinstein reduction process [45,46], a well known tool in symplectic geometry, used to reduce spaces with symmetries. Concretely, this reduction process can be summarised as follows:

Theorem 4.2.1 (Marsden–Weinstein Reduction Process)
Let (N, ω) be a symplectic manifold. Suppose that G is a Lie group acting by symplectomorphisms on (N, ω). Let $\mu : N \to \mathfrak{g}^*$ be the moment map for this action with $\mathfrak{g}$ the Lie algebra of G. Furthermore, let $c \in \mathfrak{g}^*$ be a regular value of μ and $\mathsf{G}_c \subseteq \mathsf{G}$ the (coadjoint) stabiliser group

[1] Note that the horizontal lift of X to T^*M^3 is trivial because of the assumed form of the metric $\mathring{g}_3$.

of c. We assume that G_c acts freely and properly on $\mu^{-1}(\{c\})$. Set $N_c := \mu^{-1}(\{c\})/\mathsf{G}_c$, with $\mathfrak{p} : \mu^{-1}(\{c\}) \to N_c$ the natural projection, and consider,

$$\begin{array}{ccc} \mu^{-1}(\{c\}) & \xhookrightarrow{\mathfrak{i}} & N \\ \downarrow{\scriptstyle \mathfrak{p}} & & \\ N_c & & \end{array} \tag{4.2.1}$$

Then, there exists a unique symplectic structure ω_c on N_c such that $\mathfrak{p}^*\omega_c = \mathfrak{i}^*\omega$.

Consider now the following 'twisted' symplectic form on T^*M

$$\omega_3 \;:=\; \mathrm{d}q_i \wedge \mathrm{d}x^i - \mathrm{d}(\lambda q_3) \wedge \mathrm{d}x^3 \;, \tag{4.2.2}$$

where $\lambda \in \mathscr{C}^\infty(M^2)$ is non-vanishing, and again let $\frac{\partial}{\partial x^3}$ be the infinitesimal generator of the Lie algebra corresponding to G. Evidently $\partial_{x^3} \lrcorner\, \omega = \mathrm{d}(\lambda q_3)$ and ω_3 is closed. Hence, $\mathcal{L}_{\frac{\partial}{\partial x^3}}\omega_3 = 0$ and $\frac{\partial}{\partial x^3}$ acts by symplectomorphisms on (T^*M, ω_3). Further, as $\partial_{x^3} \lrcorner\, \omega$ is exact, we can define the moment map on T^*M by $\mu(x,q) = \lambda q_3$ (up to shifts by a constant), from which it follows that $\mu^{-1}(\{c\}) = \{(x,q)\,|\, q_3 = \frac{c}{\lambda}\}$ for any regular value $c \in \mathbb{R}$. Consequently, $\mu^{-1}(\{c\})/\mathsf{G}_c$ is locally given by $(x^i, x^3, q_i, q_3) = (x^i, \mathrm{const}, q_i, q_3 = v_3(x^i))$. Furthermore, by virtue of Theorem 4.2.1, we obtain the symplectic form $\omega_c := \mathrm{d}q_i \wedge \mathrm{d}x^i$ on $\mu^{-1}(\{c\})/\mathsf{G}_c \cong T^*M^2$ satisfying $\mathfrak{p}^*\omega_c = \mathfrak{i}^*\omega_3$, as well as two closed differential two-forms given by

$$\begin{aligned} \tilde{\varpi}_2 \;&:=\; \mathrm{e}^{\varphi}\big(\varpi_2 + q_i \mathring{\nabla}^i\varphi\, \mathrm{vol}_{M^2}\big) \;, \\ \tilde{\alpha}_2 \;&:=\; \mathrm{e}^{\varphi}\big\{\alpha_2 - \big[\hat{h}_+ + \mathrm{e}^{-2\varphi}\big(\mathring{\nabla}^i\varphi\partial_i\varphi\, q_3^2 - q_3\mathring{\nabla}^i\varphi\partial_i q_3\big)\big]\,\mathrm{vol}_{M^2} + q_i\mathring{\nabla}^i\varphi\,\varpi_2\big\} \;, \end{aligned} \tag{4.2.3}$$

which are simply those from (4.1.3), with q_3 understood as a function of x^1 and x^2.

Upon requiring the vanishing of the pullback of $\tilde{\varpi}_2$ and $\tilde{\alpha}_2$ along (3.2.2), together with the relabelling the function q_3 by v_3, we obtain

$$\begin{aligned} \mathring{\nabla}_i v^i \;&=\; -v^i\partial_i\varphi \;, \\ \mathring{\Delta}_{\mathrm{B}} p + \mathring{\nabla}_i v^j \mathring{\nabla}_j v^i + \tfrac{1}{2}|v|^2\mathring{R} \;&=\; -\mathring{g}^{ij}\partial_i\varphi\partial_j p + v^i v^j \mathring{\nabla}_i\partial_j\varphi \\ &\qquad + \mathrm{e}^{-2\varphi}\big[\big(\mathring{\Delta}_{\mathrm{B}}\varphi - \mathring{g}^{ij}\partial_i\varphi\partial_j\varphi\big)v_3^2 + 2v_3\mathring{g}^{ij}\partial_i\varphi\partial_j v_3\big] \;. \end{aligned} \tag{4.2.4}$$

These are precisely the incompressibility equation (1.2.3b) and the pressure equation (1.2.3c) when adapted to the warped product metric (4.1.1) and under the assumption that p is independent of x^3. Additionally, the first equation of (4.2.4) can be rewritten as $\mathring{\nabla}^i(\mathrm{e}^{\varphi}v_i) = 0$, hence by the Poincaré lemma, any solution is locally of the form

$$v_i \;=\; -\sqrt{\det(\mathring{g}_2)}\,\mathrm{e}^{-\varphi}\varepsilon_{ij}\mathring{g}^{jk}\partial_k\psi \,, \tag{4.2.5}$$

for some $\psi \in \mathscr{C}^\infty(M^2)$. That is, we have a modified compressibility equation and modified stream function in two dimensions post reduction. Evidently, when $\varphi = 0$, we find that v_3 is unconstrained by (4.2.4), x^3 coordinatises $\mathbb{R}$, and we obtain, from our reduced flow, the standard situation of an incompressible fluid in two dimensions, as discussed in Section 2.2.

Next, let X be a vector field on $\mu^{-1}(\{c\})/\mathsf{G}_c \cong T^*M^2$ and consider its horizontal lift $\tilde{X}$ to T^*M^3 using the Levi-Civita connection for the metric (4.1.1),

$$\tilde{X} \;\coloneqq\; X + X \lrcorner\, \mathrm{d}x^i \mathring{\Gamma}_{i3}{}^3 q_3 \frac{\partial}{\partial q_3} \;=\; X + X \lrcorner\, \mathrm{d}\varphi\, q_3 \frac{\partial}{\partial q_3}\;. \tag{4.2.6}$$

Noting that $\tilde{X} \lrcorner \mathring{\nabla} q_3 = 0$ and $\varpi_2 \wedge (\alpha_2 - \hat{h}_+ \operatorname{vol}_{M^2}) = 0$, we obtain from (4.1.2) that $\alpha_3 \wedge (\tilde{X} \lrcorner \alpha_3) = -2\varpi_2 \wedge X \lrcorner (\alpha_2 - \hat{h}_+ \operatorname{vol}_{M^2}) \wedge \mathring{\nabla} q_3 \wedge \mathrm{d}x^3$. Consequently, the endomorphism (3.3.5) becomes

$$\hat{\mathcal{J}}_3 \tilde{X} \;=\; \frac{1}{\sqrt{|\hat{f}_2 + \hat{h}_+|}}\, \varepsilon_2 \lrcorner \big[\varpi_2 \wedge X \lrcorner \big(\alpha_2 - \hat{h}_+ \operatorname{vol}_{M^2}\big)\big]\;, \tag{4.2.7}$$

where ε_2 is the dual to the Liouville volume form on T^*M^2; see also Remark 4.1.1. Hence, we obtain an endomorphism $\hat{\mathcal{J}}_2$ on $\mu^{-1}(\{c\})/\mathsf{G}_c$, that is precisely of the form (3.2.5) for the Monge–Ampère structure $\big(\varpi_2, \alpha_2 - \hat{h}_+ \operatorname{vol}_{M^2}\big)$.[1] Note that $\alpha_2 - \hat{h}_+ \operatorname{vol}_{M^2}$ is simply α_2 with $\hat{f}_2$ replaced by $\hat{f}_2 + \hat{h}_+$. Note also that whilst ϖ_2 is closed, $\alpha_2 - \hat{h}_+ \operatorname{vol}_{M^2}$ is not. Mirroring (3.2.6) we set

$$\hat{\mathcal{K}}_2 \;\coloneqq\; \operatorname{sgn}(\hat{f}_2 + \hat{h}_+)\sqrt{|\hat{f}_2 + \hat{h}_+|}\, \mathring{\nabla} q_i \wedge \mathrm{d}x^i\;. \tag{4.2.8}$$

Then, as before, $\hat{\mathcal{K}}_2(\hat{\mathcal{J}}_2 X, Y) = -\hat{\mathcal{K}}_2(X, \hat{\mathcal{J}}_2 Y)$ for all vector fields X and Y on $\mu^{-1}(\{c\})/\mathsf{G}_c$ so that $\hat{g}_2(X,Y) \coloneqq \hat{\mathcal{K}}_2(X, \hat{\mathcal{J}}_2 Y)$ is an almost (para-)Hermitian metric on $\mu^{-1}(\{c\})/\mathsf{G}_c$. Explicitly,

$$\hat{g}_2 \;=\; \tfrac{1}{2}(\hat{f}_2 + \hat{h}_+)\mathring{g}_{ij}\mathrm{d}x^i \odot \mathrm{d}x^j + \tfrac{1}{2}\mathring{g}^{ij}\mathring{\nabla} q_i \odot \mathring{\nabla} q_j\;. \tag{4.2.9}$$

Let us close this section by stating the pullback of the metric (4.2.9) along

$$\iota \,:\, x^i \;\mapsto\; (x^i, q_i) \;\coloneqq\; \big(x^i, -\sqrt{\det(\mathring{g}_2)}\, \mathrm{e}^{-\varphi}\varepsilon_{ij}\mathring{g}^{jk}\partial_k\psi\big)\,, \tag{4.2.10}$$

given by combining (3.2.2) and (4.2.5):

$$g_2 \;=\; \tfrac{1}{2}\big(\mathring{\Delta}_{\mathrm{B}}\psi \mathring{\nabla}_i\partial_j\psi + T_{ij}\big)\mathrm{e}^{-2\varphi}\mathrm{d}x^i \odot \mathrm{d}x^j \tag{4.2.11a}$$

with

$$\begin{aligned} T_{ij} \;\coloneqq\; \mathring{g}_{ij}\big\{&\mathring{\nabla}^l\varphi\,\partial_l\psi\big(\mathring{\nabla}^k\varphi\,\partial_k\psi - \mathring{\Delta}_{\mathrm{B}}\psi\big) - \big(\mathring{\nabla}^k\varphi\,\partial_k\varphi\big)\big(\mathring{\nabla}^l\psi\,\partial_l\psi\big)\\ &+ \mathring{\nabla}^k\varphi\big[\mathring{\nabla}^l\psi\,\mathring{\nabla}_k\partial_l\psi + v_3\big(\partial_k v_3 - v_3\partial_k\varphi\big)\big]\big\}\\ &+ \partial_i\varphi\,\partial_j\varphi\big(\mathring{\nabla}^k\psi\,\partial_k\psi\big) - \mathring{\nabla}^k\psi\big[\partial_i\varphi\,\mathring{\nabla}_j\partial_k\psi + \partial_j\varphi\,\mathring{\nabla}_i\partial_k\psi\big]\,. \end{aligned} \tag{4.2.11b}$$

Evidently, $T_{ij} = 0$ when $\varphi = 0$, and we recovers the metric (2.2.7). It should be noted, however, that when $\varphi \neq 0$, the correspondence between the signature of g_2 and the sign of the pullback of $(\hat{f}_2 + \hat{h}_+)$ via (4.2.10) is not direct, as we shall see in Section 4.4. This observation echoes the discussion following (3.3.11) for incompressible flows in dimension higher than two.

[1] Here, $\hat{h}_+$ is understood as a function of x^1, x^2, q_1, and q_2 only, since in $\mu^{-1}(\{c\})/\mathsf{G}_c$, $q_3 = q_3(x^1, x^2)$.

Remark 4.2.2 (Non-zero γ)
It should be clear at this stage that the computations in this chapter may be extended to cases where $\gamma \not\equiv 0$ and the symmetry $\frac{\partial}{\partial x^3} + \gamma\frac{\partial}{\partial q_3}$ lies in $\mathfrak{X}(T^*M)$. In particular, given some $\gamma = \gamma(x, y)$, it is necessary (by exactness of $\mathrm{d}\mu$) for us to be able to choose non-vanishing $\lambda \in \mathscr{C}^\infty(M^2)$ such that $\gamma\lambda = d$ is constant in x^1 and x^2. It then follows that the third component of the velocity must be of the form $v_3 = \frac{1}{\lambda}(dx^3 + c)$ for $c, d \in \mathbb{R}$. For $d \neq 0$, we can set $d = 1$ without loss of generality by scaling either γ or λ, in which case the third velocity component simplifies further to $v_3 = \gamma(x^3 + \tilde{c})$, for $\tilde{c} \in \mathbb{R}$ arbitrary, while setting $d = 0$ corresponds $\gamma \equiv 0$, as discussed above. Setting $c = 0$ corresponds to the Burgers' type reduction considered in [11]. Hence, our reduction approach provides an extension of the Lundgren transformation to flows of the form (4.0.1), which satisfy either of the additional constraints $W(x^1, x^2) = \tilde{c}\gamma(x^1, x^2)$ or $\gamma \equiv 0$.

4.3 k-Plectic Reduction

We saw in Chapter 3 that k-plectic geometry is an appropriate language in which to formulate higher dimensional flows. It may then be reasonably assumed that a k-plectic generalisation of Theorem 4.2.1 would be an appropriate tool when considering symmetry reductions. Fortunately for us, [47] have recently produced such a generalisation:

Theorem 4.3.1
Let (N, ϖ) be a k-plectic manifold. Suppose that G is a Lie group acting by k-plectomorphisms on (N, ϖ). Let $\mu : N \to \bigwedge^{k-1} T^*N \otimes \mathfrak{g}^*$ be the moment map for this action with $\mathfrak{g}$ the Lie algebra of G. Furthermore, let $c \in \Omega^{k-1}(N, \mathfrak{g}^*)$ be closed and define

$$\begin{aligned} \mu^{-1}(\{c\}) &:= \{x \in N \,|\, \mu(x) = c_x\}, \\ \mathsf{G}_c &:= \big\{g \in \mathsf{G} \,\big|\, g_*^{-1}X_1 \lrcorner \ldots \lrcorner\, g_*^{-1}X_{k-1} \lrcorner\, \mathrm{Ad}^*_g c_{g^{-1}x} = X_1 \lrcorner \ldots \lrcorner\, X_{k-1} \lrcorner\, c_x \\ &\qquad \text{for all } x \in N \text{ and for all } X_1, \ldots, X_{k-1} \in T_xN\big\}. \end{aligned} \tag{4.3.1}$$

Suppose that $\mu^{-1}(\{c\})$ is a submanifold of N with (smooth) embedding $\mathfrak{i} : \mu^{-1}(\{c\}) \hookrightarrow N$ and that G_c acts freely and properly on $\mu^{-1}(\{c\})$. Set $N_c := \mu^{-1}(\{c\})/\mathsf{G}_c$, again with natural projection $\mathfrak{p} : \mu^{-1}(\{c\}) \to N_c$ and consider,

$$\begin{array}{ccc} \mu^{-1}(\{c\}) & \xrightarrow{\ \mathfrak{i}\ } & N \\ \downarrow \mathfrak{p} & & \\ N_c & & \end{array} \tag{4.3.2}$$

Then, there exists a unique closed differential form $\varpi_c \in \Omega^{k+1}(N_c)$ on N_c such that $\mathfrak{p}^*\varpi_c = \mathfrak{i}^*\varpi$.

Evidently, for $k = 1$ this result reduces to Theorem 4.2.1. It is important to stress that for $k > 1$, the differential form $\varpi_c \in \Omega^{k+1}(M_c)$ might be degenerate.

Consider now the 2-plectic reduction of the Monge–Ampère structure (3.3.1) via Theorem 4.3.1, recalling (4.1.5), which implies Lie groups corresponding to the infinitesimal generator $\frac{\partial}{\partial x^3}$ act by k-plectomorphism on the 2-plectic manifold $(T^*M, \varpi_3, \alpha_3)$. Further, by virtue of exactness (4.1.4), the moment map may be defined (up to a shift by an exact form) by

$$\mu(x,q) \;=\; \star_{\mathring{g}_2} \mathrm{e}^{\varphi} q_i \mathrm{d}x^i \,. \tag{4.3.3}$$

For $\psi \in \mathscr{C}^\infty(M^2)$, $\mu^{-1}(\{-\mathrm{d}\psi\})$ is non-empty and given by

$$\mu^{-1}(\{-\mathrm{d}\psi\}) \;=\; \Big\{(x,q)\,\Big|\, q_i = -\sqrt{\det(\mathring{g}_2)}\,\mathrm{e}^{-\varphi}\varepsilon_{ij}\mathring{g}^{jk}\partial_k\psi\Big\} \,. \tag{4.3.4}$$

Consequently, the quotient $\mu^{-1}(\{-\mathrm{d}\psi\})/\mathsf{G}_{-\mathrm{d}\psi}$ is locally given by $(x^i, x^3, q_i, q_3) = \big(x^i, \mathrm{const}, -\sqrt{\det(\mathring{g}_2)}\,\mathrm{e}^{-\varphi}\varepsilon_{ij}\mathring{g}^{jk}\partial_k\psi, q_3\big)$. Furthermore, there exists closed differential form $\varpi_{-\mathrm{d}\psi} := \mathrm{e}^{-\varphi}\mathrm{vol}_{M^2} \wedge \mathrm{d}q_3$ on $\mu^{-1}(\{-\mathrm{d}\psi\})/\mathsf{G}_{-\mathrm{d}\psi}$ satisfying $\mathfrak{p}^*\varvarpi_{-\mathrm{d}\psi} = \mathfrak{i}^*\varpi_3$. Returning to (4.1.3), while the pullback of ϖ'_2 to $\mu^{-1}(\{-\mathrm{d}\psi\})$ vanishes identically, the equation $\mathfrak{p}^*\alpha_{-\mathrm{d}\psi} = \mathfrak{i}^*\alpha'_2$ is satisfied by the two-form

$$\alpha_{-\mathrm{d}\psi} \;:=\; \mathrm{e}^{\varphi}\Big[\det\Big(\mathring{\nabla}^i q_j\Big) - \big(\hat{f}_2 + \hat{h}_-\big)\Big]\Big|_{q_i = -\sqrt{\det(\mathring{g}_2)}\,\mathrm{e}^{-\varphi}\varepsilon_{ij}\mathring{g}^{jk}\partial_k\psi}\mathrm{vol}_{M^2} + \mathrm{e}^{-\varphi}q_3\mathrm{d}q_3 \wedge \star_{\mathring{g}_2}\mathrm{d}\varphi \tag{4.3.5}$$

on $\mu^{-1}(\{-\mathrm{d}\psi\})/\mathsf{G}_{-\mathrm{d}\psi}$, where $\hat{h}_-$ is as defined in (4.1.2b). Finally, requiring that the pullback of $\alpha_{-\mathrm{d}\psi}$ along

$$\iota \,:\, x^i \;\mapsto\; (x^i, q_3) \;:=\; (x^i, v_3(x^i)) \,, \tag{4.3.6}$$

vanishes, the second equation of (4.2.4), with v_i given by $v_i = -\sqrt{\det(\mathring{g}_2)}\,\mathrm{e}^{-\varphi}\varepsilon_{ij}\mathring{g}^{jk}\partial_k\psi$, is obtained, trivially satisfying the first equation from (4.2.4), as discussed around (4.2.5). Observe that, in contrast to the symplectic reduction where Poincaré's lemma is required, the 2-plectic reduction of the Monge–Ampère structure directly yields a two-dimensional flow (which may not be incompressible) in terms of a stream function ψ, with the trade off that the metric (4.2.9) is not acquired. As a result, the k-plectic reduction is a more elegant, compact tool, should a description of the reduced kinematics be all that is required.

4.4 Reducible Incompressible Fluids - Examples

In this section we provide a pair of examples of flows on which one may apply symplectic and 2-plectic reductions - one with the symmetry Lie group given by $\mathbb{R}$ and the other with Lie group given by S^1.

4.4.1 Arnold–Beltrami–Childress (ABC) Flows

Let us consider flows on $M_3 := \mathbb{R}^3$ equipped with the standard Euclidean metric

$$\mathring{g}_3 \;:=\; \mathring{g}_2 + \mathrm{d}z \otimes \mathrm{d}z \quad \text{with} \quad \mathring{g}_2 \;:=\; \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y \,, \tag{4.4.1}$$

which corresponds to (4.1.1) with $\varphi = 0$. Note also that $\hat{h}_\pm = 0$. In this case, the infinitesimal generator $\frac{\partial}{\partial x^3} = \frac{\partial}{\partial z}$ corresponds to the choice $\mathsf{G} = \mathbb{R}$ acting by translation in z. Our symplectic reduction therefore yields an incompressible fluid flow in two dimensions, on Euclidean background. In summary, the equations (4.2.3) reduce to $\tilde{\varpi}_2 = \varpi_2$, and $\tilde{\alpha}_2 = \alpha_2$, with the incompressibility and pressure equations (4.2.4) respectively given by

$$\partial_x v_x + \partial_y v_y \ = \ 0 \tag{4.4.2a}$$

and

$$\Delta p \ = \ 2\big(\partial_x v_x \partial_y v_y - \partial_x v_y \partial_y v_x\big) \quad \text{with} \quad \Delta \ := \ \partial_x^2 + \partial_y^2 \ , \tag{4.4.2b}$$

where v_x and v_y are functions of x and y only.

Additionally, performing the 2-plectic reduction, or equivalently applying Poincaré lemma to (4.4.2a), yields the velocity components v_x and v_y

$$q_x \ := \ v_x \ = \ -\partial_y \psi \quad \text{and} \quad q_y \ := \ v_y \ = \ \partial_x \psi \,, \tag{4.4.3}$$

in terms of a stream function $\psi = \psi(x,y)$. The differential form corresponding to (4.3.5) is given by

$$\alpha_{-\mathrm{d}\psi} \ = \ \big[\partial_x^2 \psi\, \partial_y^2 \psi - (\partial_x \partial_y \psi)^2 - \tfrac{1}{2}\Delta p\big]\, \mathrm{d}x \wedge \mathrm{d}y \ , \tag{4.4.4}$$

which is unchanged when pulled back along $(x,y) \mapsto (x,y,q_z) := (x,y,v_z(x,y))$. Hence, imposing a vanishing pullback condition is equivalent to the Monge–Ampère equation

$$\tfrac{1}{2}\Delta p \ = \ \partial_x^2 \psi\, \partial_y^2 \psi - (\partial_x \partial_y \psi)^2 \ , \tag{4.4.5}$$

which is, in turn, precisely (4.4.2b) with v_x, v_y evaluated as per (4.4.3). Hence, upon making the free choice of a pair of z-independent functions ψ and v_z, an incompressible fluid flow in $\mathbb{R}^3$ that reduces to an incompressible flow on the (x,y)-plane is recovered.

Making the choice

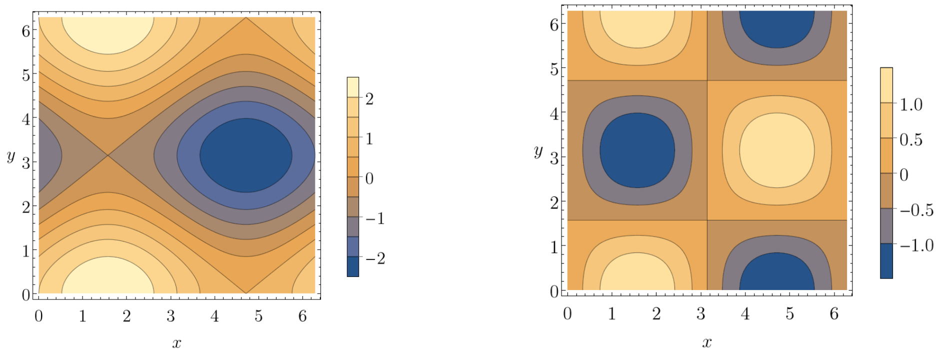
$$v_z(x,y) \ \equiv \ \psi(x,y) \ := \ A\cos(y) + B\sin(x) \,, \tag{4.4.6}$$

for $A, B \in \mathbb{R}$ some constants (see Figure 4.4.1a), and computing (4.4.3), we recover the velocity field for the integrable case of Arnold–Beltrami–Childress (ABC) flow [48],

$$(v_x, v_y, v_z) \ = \ (\dot{x}, \dot{y}, \dot{z}) \ = \ \big(A\sin(y), B\cos(x), A\cos(y) + B\sin(x)\big) \,. \tag{4.4.7}$$

Following [48], upon taking the quotient of v_x and v_y, this system integrates to $v_z = A\cos(y) + B\sin(x) = \text{const}$. Furthermore, (4.4.5) becomes

$$\hat{f}_2 + \hat{h}_+ \ = \ \tfrac{1}{2}\Delta p \ = \ AB\sin(x)\cos(y) \ , \tag{4.4.8}$$



(a) Streamlines for ψ. Note $\psi = 0$ corresponds to vanishing vorticity and defines a shear layer between two homoclinic orbits.

(b) Contour plot for $\hat{f}_2 + \hat{h}_+$. Note the domain is partitioned into squares of side length π, across which Δp alternates sign.

Figure 4.4.1: Plots of the iso-lines of the stream function (4.4.6) and reduced Laplacian of pressure (4.4.8) for Arnold–Beltrami–Childress flows with parameters $A = 1.5$ and $B = 1$.

as displayed in Figure 4.4.1b. Since $M^2 = \mathbb{R}^2$ and $\hat{h}_+ = 0$, it follows that the metric (4.2.9) on the reduced phase space $\mu^{-1}(\{c\})/\mathsf{G}_c \cong T^*\mathbb{R}^2$ is precisely (2.3.2). Hence, we may follow exactly the treatment from Section 2.3 and the curvature scalar $\hat{R}_2$ for the metric (4.2.9) follows directly from (2.3.3),

$$\hat{R}_2 \;=\; \frac{\sin^2(x) + \cos^2(y)}{AB\sin^3(x)\cos^3(y)}\,, \tag{4.4.9}$$

and, as in previous examples, for $\hat{f}_2 \gtrless 0$ the metric $\hat{g}_2$ is Riemannian/Kleinian with positive/negative scalar curvature. Again, when $\hat{f}_2 = 0$, both the metric and the curvature scalar are singular.

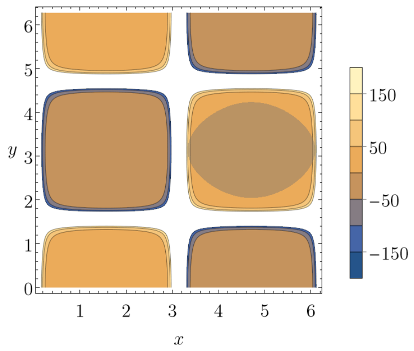
In turn, the pullback metric (4.2.11), with v_x and v_y as given in (4.4.7), is

$$(g_{2\,ij}) \;=\; [A\cos(y) + B\sin(x)]\begin{pmatrix} B\sin(x) & 0 \\ 0 & A\cos(y)\end{pmatrix}, \tag{4.4.10}$$

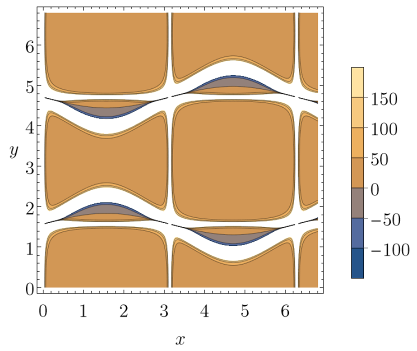
where the vorticity in two dimensions is $\zeta := \Delta\psi = -\psi = -A\cos(y) - B\sin(x)$. This metric is singular when $\hat{f}_2 = 0$, with a further singularity when $A\cos(y) + B\sin(x) = 0$, precisely along the shear layer featuring in Figure 4.4.1a corresponding to vorticity vanishing. The curvature scalar R_2 associated with (4.4.10) is then

$$R_2 \;=\; \frac{B\sin(x)\big[\sin^2(x) + 3\cos^2(y)\big] + A\cos(y)\big[\cos^2(y) + 3\sin^2(x)\big]}{2\sin^2(x)\cos^2(y)[B\sin(x) + A\cos(y)]^3}\,. \tag{4.4.11}$$

Observe that the lines $x = n\pi$ and $y = \big(n + \frac{1}{2}\big)\pi$ for all $n \in \mathbb{Z}$, along which $f_2 = 0$, are singularities of both the metric g and its curvature R, as was the case for the metric (4.2.9).



(a) Contours of the curvature scalar $\hat{R}_2$. Note the signs of $\hat{R}_2$ and $\hat{f}_2$ agree and $\hat{R}_2$ blows up as $\hat{f}_2$ tends to zero.



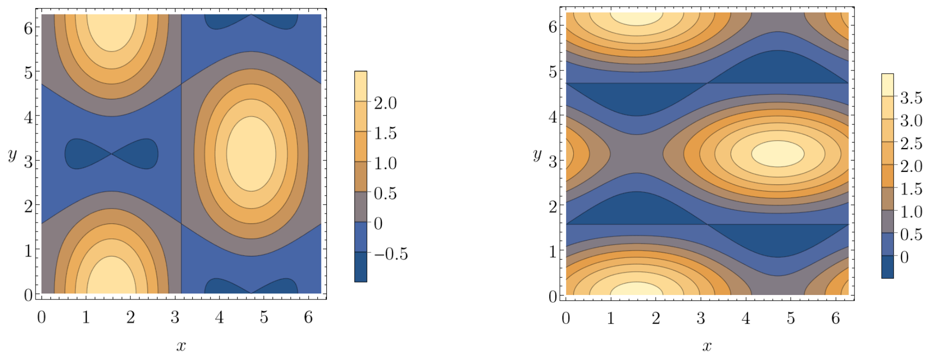
(b) Contours of R_2. Note $\hat{f}_2$ is not a diagnostic for the sign of R_2, but curvature singularities still occur when $\hat{f}_2 = 0$.

Figure 4.4.2: Contour plots of the curvatures (4.4.9) and (4.4.11) respectively, for the integrable ABC flow with parameters $A = 1.5$ and $B = 1$. The ellipse highlighted on the left is the domain in M bounded by the closed streamline $\psi = -\frac{27}{16}$, contained in a region on which the metrics $\hat{g}_2$ and g are Riemannian, and $\hat{f}_2 > 0$, as discussed around (2.2.16).

Additionally, the presence of $A\cos(y) + B\sin(x)$ in the denominator illustrates that the shear layer is in fact a curvature singularity. See Figure 4.4.2b. This curvature singularity arises due to the vanishing vorticity and is otherwise unseen by the pressure criterion. Furthermore, g_2 is globally degenerate when $A = 0$ or $B = 0$ independently, in addition to when both A and B vanish (in which case both the vorticity and the Hessian part of the metric vanish). Whilst the latter corresponds to when $\psi = 0$ and there is no flow, the former two choices seem to correspond to cases where the flow is trivial as a result of further symmetry: let $B = 0$ (analogously for $A = 0$), such that the stream function (4.4.6) depends only on y (respectively x) and the flow velocity (4.4.7) has only two non-zero components, which in turn depend only on y (respectively x). It follows that the streamlines are simply $y =$ constant (analogously $x =$ constant) and (as we may still reduce along the z-axis), we have one-dimensional flow in the (x, y)-plane. Recall that we saw similar behaviour in Section 2.3.2 when global degeneracy occurred.

4.4.2 Hill's Vortex Revisited

Let us return to considering flows on background manifold given by $M = \mathbb{R}^2 \times_{r^2} S^1$ and recall that the metric on M given by (3.4.1) induces standard cylindrical coordinates, where $r \in \mathbb{R}^+$, $z \in \mathbb{R}$, and $\theta \in [0, 2\pi)$. In this case, the infinitesimal generator $\frac{\partial}{\partial x^3} = \frac{\partial}{\partial \theta}$ corresponds to teh choice $G = S^1$ acting by translation in θ, modulo 2π. Then, assuming the pressure is given by $p = p(r, z)$, we have $\varphi = \log(r)$ and $\hat{h}_+ = \frac{1}{2r}\partial_r p$ in (4.1.2). Hence, the differential forms (4.2.3)



(a) Contours of the eigenvalue E_+. In addition to the shear layers, note that $E_+ = 0$ on the vertical $x = \pi$.

(b) Contour plot for the eigenvalue E_-. In addition to the shear layers, E_- also vanishes along $y = \frac{\pi}{2}$ and $y = \frac{3\pi}{2}$.

Figure 4.4.3: Plots of the eigenvalues of the pullback metric $g_2 = \mathrm{diag}(E_+, E_-)$ from (4.4.10), for the ABC flow with parameters $A = 1.5$ and $B = 1$. Note that g_2 remains Kleinian on either side of the shear layers, on which vorticity changes sign, as the two eigenvalues swap signs.

reduce to

$$\begin{aligned} \tilde{\varpi}_2 &= r\big(\varpi_2 + \tfrac{1}{r} q_r \,\mathrm{d}r \wedge \mathrm{d}z\big) , \\ \tilde{\alpha}_2 &= r\big\{\alpha_2 - \big[\tfrac{1}{2r}\partial_r p + \tfrac{1}{r^2}\big(\tfrac{1}{r^2} q_\theta^2 - \tfrac{1}{r} q_\theta \partial_r q_\theta\big)\big] \,\mathrm{d}r \wedge \mathrm{d}z + \tfrac{1}{r} q_r \varpi_2\big\} . \end{aligned} \tag{4.4.12}$$

Furthermore, the requirement of the vanishing of the pullbacks of $\tilde{\varpi}_2$ and $\tilde{\alpha}_2$ under (3.2.2) become

$$\tfrac{1}{r}\partial_r(r v_r) + \partial_z v_z \;=\; 0 \;, \tag{4.4.13a}$$

and

$$\tfrac{1}{r}\partial_r(r\partial_r p) + \partial_z^2 p \;=\; 2\big[\partial_r v_r \partial_z v_z - \partial_r v_z \partial_z v_r - \tfrac{1}{r^2} v_r^2 - \tfrac{1}{r^4}\big(v_\theta^2 - \tfrac{r}{2}\partial_r v_\theta^2\big)\big] , \tag{4.4.13b}$$

which are the equations (4.2.4) for the metric (3.4.1), with $v_\theta = v_\theta(r, z)$ arbitrary. Evidently equations (4.4.13) are precisely (3.4.3), under the assumption that the velocity and pressure are θ-independent; in particular, note that the left hand side of (4.4.13a) is simply the divergence of such a v and the left hand side (4.4.13b) is the Laplacian of $p = p(r, z)$, both expressed in cylindrical coordinates.

Turning to the 2-plectic reduction, note that the moment map (4.3.3) can be taken as

$$\mu(x, q) \;=\; r q_r \mathrm{d}z - r q_z \mathrm{d}r \;, \tag{4.4.14}$$

from which it follows that, locally on $\mu^{-1}(\{-\mathrm{d}\psi\})/\mathsf{G}_{-\mathrm{d}\psi}$, must have

$$q_r \;:=\; v_r \;=\; -\tfrac{1}{r}\partial_z \psi \quad \text{and} \quad q_z \;:=\; v_z \;=\; \tfrac{1}{r}\partial_r \psi \;, \tag{4.4.15}$$

which can be interpreted as expressions for the velocity components in the r and z directions, in terms of a stream function $\psi = \psi(r,z)$ in two dimensions. In fact, imposing that the pullback of the closed differential form (4.3.5) along $(r,z) \mapsto (r,z,q_\theta) := (r,z,v_\theta(r,z))$ vanishes, we find

$$\begin{aligned} \tfrac{1}{2}\big[\tfrac{1}{r}\partial_r(r\partial_r p) + \partial_z^2 p\big] &= \tfrac{1}{r^2}\big[\partial_r^2\psi\partial_z^2\psi - (\partial_r\partial_z\psi)^2\big] - \tfrac{1}{r^4}(\partial_z\psi)^2\ , \\ &\quad + \tfrac{1}{r^3}\big(\partial_z\psi\partial_r\partial_z\psi - \partial_r\psi\partial_z^2\psi\big) - \frac{1}{r^4}\Big(v_\theta^2 - \frac{r}{2}\partial_r v_\theta^2\Big) \end{aligned} \tag{4.4.16}$$

that is, (4.4.13b) with v_r and v_z given in terms of ψ as in (4.4.15). Observe that there is freedom to choose any θ-independent ψ and v_θ, provided they satisfy (4.4.16). Further, the adapted incompressibility equation (4.4.13a) is trivially satisfied for any such choices, given (4.4.15).

In what follows, we fix of ψ and v_θ corresponding to the interior of Hill's spherical vortex, as in Section 3.4.1. Namely, we set $v_\theta = 0$ and restate the stream function:

$$\psi(r,z) \ := \ \tfrac{3}{4}r^2\left(r^2 + z^2 - 1\right)\ . \tag{4.4.17}$$

Again, the remaining velocity components are given by (3.4.7). Note that these components being of the form (3.4.4) can now be seen as an automatic consequence of the symmetry, as opposed to being a prescription like they were in our previous analysis. Following from (4.4.16) the Laplacian of pressure is given by[1]

$$\hat{f}_2 + \hat{h}_+ \ := \ \tfrac{1}{2}\big(\partial_r^2 p + \partial_z^2 p + \tfrac{1}{r}\partial_r p\big) \ = \ \tfrac{9}{4}\big(4r^2 - 3z^2\big)\,, \tag{4.4.18}$$

which is precisely the function (3.4.8) from our three dimensional analysis (See also Figure 3.4.1b).

The metric (4.2.9) then takes the form

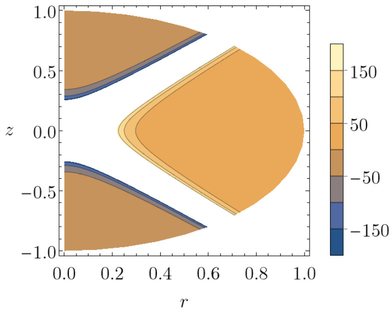
$$(\hat{g}_{ij}) \ = \ \operatorname{diag}\left[\left(\hat{f}_2 + \hat{h}_+\right)\delta_{ij}, \delta^{ij}\right] \tag{4.4.19}$$

and $\hat{R}_2$ is given by (2.3.3), with f replaced by $\hat{f}_2 + \hat{h}_+$. Namely

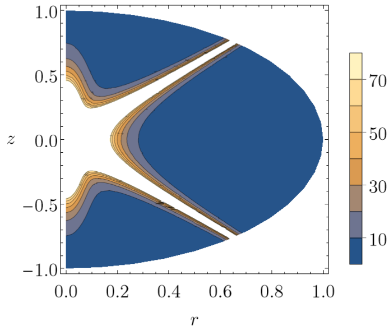
$$\hat{R}_2 \ = \ \frac{56\big(4r^2 + 3z^2\big)}{9\big(4r^2 - 3z^2\big)^3}\ . \tag{4.4.20}$$

See Figure 4.4.4a. Note that when $4r^2 > 3z^2$, coinciding with $\hat{f}_2 + \hat{h}_+ > 0$, (4.4.19) is Riemannian and the curvature scalar is positive. Similarly, the metric is Kleinian and the curvature scalar negative when $\hat{f}_2 + \hat{h}_+ < 0$ and $4r^2 < 3z^2$. Furthermore, the metric is singular when $4r^2 = 3z^2$, that is, when $\hat{f}_2 + \hat{h}_+ = 0$ and it is clear that this singularity is also one for the curvature. This mostly coincides with what we saw in our genuine three-dimensional analysis Section 3.4.1, however the sign of the curvature no-longer varies on the Kleinian region.

[1] Note here that $\hat{f}_2 + \hat{h}_+$ has no dependence on q_i, q_θ, or θ itself, hence is unchanged under the pullback (3.2.2).



(a) Plot of $\tilde{R}_2$. The curvature is singular along $4r^2 = 3z^2$ and is negative at the front and back of the sphere (with respect to background flow in the z direction).



(b) Plot of R_2. The curvature is is singular along $100r^4 - 71r^2z^2 - 2z^4$, in-line with where $E_- = 0$ and hence where the metric g_2 is degenerate.

Figure 4.4.4: Respective contour plots of the curvature scalars (4.4.20) and (4.4.22) for the interior of Hill's vortex. Note how both curvatures decrease in magnitude towards the boundary of the sphere.

Furthermore, the pullback metric (4.2.11) becomes

$$(g_{2\,ij}) \;=\; \frac{9}{4}\begin{pmatrix} 20r^2 - 2z^2 & 9rz \\ 9rz & 5r^2 + z^2 \end{pmatrix}, \tag{4.4.21}$$

which is precisely the top left 2×2 block from the metric (3.4.11). Hence, the eigenvalues of (4.4.21) are given by $E_\pm$ from (3.4.12), see also Figure 3.4.2. Note that as E_+ is non-negative and vanishes only at the origin, the signature of (4.4.21) is dictated precisely by the sign of E_-. Hence, g_2 is Riemannian precisely when $100r^4 - 71r^2z^2 - 2z^2 > 0$, Kleinian when $100r^4 - 71r^2z^2 - 2z^4 < 0$, and singular where there is equality. This agrees with our earlier observations around (3.4.11), however the singularity $2r^2 - z^2 = 0$, across which the Kleinian signature changes in three dimensions, does not feature. The curvature scalar R_2 associated with (4.4.21) is then given by

$$R_2 \;=\; \frac{28\big(50r^4 + z^4\big)}{9\big(100r^4 - 71r^2z^2 - 2z^4\big)^2}\,. \tag{4.4.22}$$

Observe that R_2 is everywhere positive and is singular precisely where E_- vanishes, that is, where the metric g_2 is singular. Unlike in the genuine three-dimensional computation, the sign of the curvature remains positive on the whole domain here. Additionally, observe that the region on which $\hat{f}_2 + \hat{h}_+ > 0$ still falls within the Riemannian region with respect to (4.4.21). It therefore appears that the reduced flow in two dimensions captures all of the apparently essential information from the three-dimensional computation, with respect to changes in the dominance of vorticity and strain, the sign of the Laplacian of pressure type quantities, and when the Lychagin–Rubtsov metric and its pullback are Riemannian.

4.5 Chapter Summary

In this chapter, we considered a large subclass of incompressible two-and-a-half-dimensional flows (4.0.1) in three dimensions, whose symmetry allows them to be reduced to a two-dimensional Navier–Stokes flow. Using the symplectic and 2-plectic Marsden–Weinstein reduction processes, we set up a scheme for reductions via one-dimensional Lie groups, extending the results of [44] to flows more general than those of Burgers' type. As in earlier chapters, an almost (para-)complex structure (4.2.7), almost (para-)Hermitian form (4.2.8), and a Lychagin–Rubtsov type metric (4.2.9) were induced on the reduced manifold. We demonstrated via formulæ and examples that the resulting two-dimensional flows are only incompressible in special cases and noted that, when they are not incompressible, the sign of our diagnostic function $\iota^*(\hat{f}_2 + \hat{h}_+)$, with ι given by (4.2.10), does not in general coincide precisely with the signature of the pullback of the Lychagin–Rubtsov metric on the reduced space (4.2.11).

5

Conclusions and Outlook

5.1 Report Summary

In this report we provided a partial treatment of the covariant, incompressible Navier–Stokes equations on an m-dimensional Riemannian background, through the lens of Monge–Ampère geometry. In particular, we demonstrated that the Navier–Stokes equations are encoded in the Monge–Ampère type structure (3.3.1) and that classical solutions of the corresponding Monge–Ampère type equation are given by a class of Lagrangian submanifolds of the cotangent bundle, with respect to said structure. It was also noted that the Laplacian of pressure could be given in terms of the vorticity and strain of the flow, as well as the Ricci tensor of the underlying manifold.

Specifying to two dimensions, we demonstrated that (1.2.3c) is a genuine Monge–Ampère equation for a stream function on M, with Monge–Ampère structures given by (2.2.2) and (3.3.1). Furthermore, it was shown that the Lychagin–Rubtsov metric (2.2.5) on T^*M is almost (para-)Hermitian and that its pullback (2.2.7) via local sections $\mathrm{d}\psi : U \subseteq M \to T^*M$ (or equivalently via (3.2.2)) provides the following generalisation of Weiss's criterion to incompressible flows on two-dimensional Riemannian manifolds:

> *The vorticity term in (1.2.7) dominates precisely when $f > 0$, the Monge–Ampère equation is of elliptic type, and the pullback of the Lychagin–Rubtsov metric is Riemannian. Similarly, the strain term dominates when $f < 0$, the Monge–Ampère equation for the pressure is hyperbolic, and the pullback metric is either Kleinian, or degenerate (corresponding to when $\zeta_{ij} = 0$)*

In higher dimensions, one may still define the Lychagin–Rubtsov metric and its pullback; in-fact, the pullback metric may be written in terms of the velocity gradient tensor, as in (3.3.10). It is straightforward to see that this does not, in general, simplify to Hessian form in dimension $m \neq 2$, since our flow is given locally in terms of a stream form (1.2.8), in contrast to the two-dimensional case where one has a stream function and (2.2.1) is a genuine Monge–Ampère equation on open, contractible sets. Hence the correspondence between the signature of the pullback metric and the sign of f is not yet clear. It appears that an appropriate statement to prove going forward would be akin to 'f is non-negative only within regions where the pullback metric is Riemannian,' but until this is confirmed, higher dimensional flows should be treated on a case by case basis.

We also noted that one may infer topological information about the flow from our geometric constructions. In particular, in two dimensions, recall that the curvature (2.2.13a) of the pullback metric is a function of the velocity and gradients of vorticity and strain, hence, the local Gauß–Bonnet theorem (2.2.14) allows one to relate the Euler number of a (compact) region in M to the velocity, vorticity, and strain of the flow over said domain. In three dimensions, one has to look for alternative approaches, one of which is given by the helicity (3.3.12).

In the previous chapter, we demonstrated that for three-dimensional, incompressible fluid flows on a Riemannian manifold with underlying symmetry described by a one-dimensional Lie group, a symmetry reduction may be used to produce two-dimensional flows satisfying 'adapted' Navier–Stokes equations (4.2.4). We noted that, in general, the resultant flow may not be incompressible and that, when the twisting parameter $\varphi \neq 0$ in (4.1.1), the sign of the pullback of $\hat{f}_2 + \hat{h}_+$ via (4.2.10) does not directly correlate with the signature of the pullback of the reduced Lychagin–Rubtsov metric (4.2.11), mirroring our our observations in higher dimensions (see Section 3.4.1 and Section 4.4.2 for example). More positively, our approach to symmetry reduction provided one significant by-product — an extension of the transformation of Lundgren, originally replicated using Monge–Ampère geometry in [11], to a more general subset of the two-and-a-half-dimensional flows (4.0.1).

5.2 Outlook: Classification Problems

As the results outlined above are intended to form an initial framework for the study of the Navier–Stokes equations utilising geometric techniques, there are, quite reasonably, gaps in our understanding to fill. We present below a brief outline of several of these, which shall become the focus of future study.

Let us begin by addressing the ambiguous nature of the result Proposition 2.1.6. It was previously noted that the original statement of the result by Banos [10] did not provide a strict definition for the concept of a Lagrangian submanifold being locally-a-section, which we extrapolated to mean Definition 2.1.5. In doing so we cover L with 'nice' neighbourhoods V_y which look like a section over some open subset $U_y \subseteq M$. Inspired by the definition of the local section of a bundle requiring a choice of covering, we propose that by choosing a different collection of 'nice' neighbourhoods to cover L with, we may construct a hierarchy of two-dimensional Lagrangian submanifolds, defined in terms of local sections of T^*M over M. In particular, as a local diffeomorphism Definition 2.1.4 need not be injective or surjective onto its codomain, we expect at least two stronger notions than Definition 2.1.5, corresponding to when $\pi|_L$ is injective or surjective. In particular, when applied to Lagrangian submanifolds which are also generalised solutions, one of these definitions should coincide with the concept of a classical/regular solution of a Monge–Ampère equation.

Additionally, it is known [49] that generalised solutions $\iota : L \hookrightarrow T^*M$ to Monge–Ampère structures on the cotangent bundle may exhibit singular behaviour in the form of points where the projection $\pi|_L : \pi \circ \iota : L \to M$ (with $\pi : T^*M \to M$, as ever, denoting the canonical projection) is non-immersive. Over a two-dimensional manifold M, such non-immersive points can arise as cusped/folded edges in the generalised solution [49, 50] and, in the context of fluid dynamics, have been observed in atmospheric models featuring abrupt changes in dynamics, for example, when a shock-wave occurs [16,51]. If the classification proposed above also accounts for such non-immersive points of the projection, it would then be possible to construct generalised solutions to two-dimensional Monge–Ampère structures corresponding to physical, non-classical solutions of a Monge–Ampère equation (for example weak solutions) either by defining (L, ι) such that $\pi|_L$ has the desired properties, or by patching together known (potentially multi-valued, non-immersive) solutions. This is in stark contrast to the current approach, which relies on the concept of a generating function defined locally on the cotangent bundle [52].

Recall that, given a Monge–Ampère structure (ω, α) on a four-dimensional manifold N, with an almost (para-)complex structure defined in the manner of (2.2.3), the Lychagin–Rubtsov theorem [15] states that if the almost (para-)complex structure were integrable, then the Monge–Ampère equation given by (2.1.1) is symplectically equivalent to either the Laplace equation or the wave equation. Banos [10] later showed that, given a (symplectic) Monge–Ampère equation on a six-dimensional manifold N, a similar result could be obtained by considering so-called nearly (para-)Calabi–Yau structures. In particular, consider the Monge–Ampère structure consisting of a pair (ω, α), with ω a symplectic 2-form and α an effective, non-degenerate 3-form, and define an almost (para-)complex form $\hat{\mathcal{J}}$ via the Hitchin endomorphism (3.3.4), as in (3.3.5). The quintuple given by $(N, \omega, \alpha, \hat{\mathcal{J}}, \hat{g})$, with $\hat{g}$ the Lychagin–Rubtsov metric defined up to scaling by $\hat{g}(X, Y) = \omega(\hat{\mathcal{J}}(X), Y)$ (i.e. (3.3.6)) and α Hitchin decomposable [9] (see also [8] in the context of the Navier–Stokes equations), then defines a nearly (para-)Calabi–Yau structure. If and only if this structure is also integrable and $\hat{g}$ is flat does it follow that the corresponding

Monge–Ampère equation is symplectically equivalent to one of the following three equations:

$$\begin{cases} \mathrm{Hess}(\psi) = 1 \\ \mathrm{Hess}(\psi) - \Delta(\psi) = 0 \\ \mathrm{Hess}(\psi) + \Box(\psi) = 0 \end{cases} \tag{5.2.1}$$

This leads to the following key question: what integrable geometry can be defined on our $(m-1)$-plectic Monge–Ampère type structures — structures consisting of a pair of $(m-1)$-plectic forms in $2m$ dimensions, such as (3.3.1)? In turn, given such an integrable geometry, is there a corresponding classification of our $(m-1)$-plectic Monge–Ampère type structures, in the spirit of [15, 10]? It turns out that a potential resolution to this question comes again from the work of Banos. Recall that a generalised almost complex structure on a manifold N is an endomorphism $\mathbb{J} : TN \oplus T^*N \to TN \oplus T^*N$ on the generalised tangent bundle, such that $\mathbb{J}^2 = -1$. In [53] it was noted that elements of the class of Monge–Ampère equations of divergent type in two dimensions, with Monge–Ampère structure denoted by (ω, α), each correspond to a generalised almost complex structure given by [54]

$$\mathbb{J} = \begin{pmatrix} \hat{J} & \omega^{-1} \\ -\omega - \hat{J}^2 \lrcorner\, \omega & -\hat{J}^* \end{pmatrix} \tag{5.2.2}$$

where $\hat{J}$ is the endomorphism defined by $\alpha = \hat{J} \lrcorner\, \omega$[1], J^* is its dual, and ω^{-1} the (symplectic) inverse of ω. In fact, as one may choose α to be closed without loss of generality for divergent Monge–Ampère equations, it follows that (ω, α) is a Hitchin pair [53, 55] and the generalised almost complex structure is integrable. Through the use of generating functions and the work of [56], such divergent Monge–Ampère equations are presented in [53] as 'generalised Laplace equations.' Of course, for $(m-1)$-plectic Monge–Ampère type structures on $2m$-dimensional manifold N, the above is just the case $m = 2$; to generalise this construction to an arbitrary m, we shall move to the realm of extended generalised complex geometry [57]. Similarly to how a symplectic form on N may be interpreted as a mapping from TN to T^*N, a k-plectic form may be viewed as a mapping from TN to $\bigwedge^{k-1} T^*N$. We may then define the extended generalised tangent bundle $TN \oplus \bigwedge^{k-1} T^*N$ on which we wish to define some 'extended generalised almost complex structure.'

The hope is that by classifying our $(m-1)$-plectic Monge–Ampère type structures in this manner, we may map, via k-plectomorphism, the pressure equation (1.2.3c) for example, to a simpler partial differential equation, which may then be solved, with any degeneracies being picked up by our mapping. Note also that the type of structure (5.2.2) no longer depends on

[1] By careful rescaling, this may be chosen to be the almost (para-)complex structure given in (2.2.3).

the signature of the metric $\hat{g}$, hence this type change must be encoded in $\mathbb{J}$ in some manner. In the context of the Navier–Stokes equations, one would then be able to analyse regions with vorticity/strain dominating, based on how this type change manifests. Moving our analysis into the realm of extended generalised complex geometry also affords us a wider range of techniques with which to probe our fluid dynamical models.

A

Locally-a-section Lagrangian Submanifolds

In this chapter, we shall fulfil our promise from Section 2.1 to provide a full proof of the statement Proposition 2.1.6 of Banos [10]. Recall from Definition 2.1.5 that a function $h : L \to M$ is a local diffeomorphism if there is an open neighbourhood V around each point $y \in L$ such that the restriction of h to V is a diffeomorphism onto the image $h(V)$. We also wish to restate the definition of a locally-a-section submanifold here:

Definition A.0.1 (Locally-a-section Submanifolds)
A submanifold $\iota : L \hookrightarrow T^*M$ is called locally-a-section if, for all $y \in L$ there exists $V_y \subseteq L$ an open neighbourhood of y, $U_y \subseteq M$ open, and $\psi \in \mathscr{C}^\infty(U_y)$ such that $\iota(V_y) = \mathrm{d}\psi(U_y)$.

Let M be an m-dimensional, connected, Riemannian manifold with good cover and denote the cotangent bundle by T^*M and the canonical projection by $\pi : T^*M \to M$. Let x^i and q_i, with $i = 1, \dots m$, denote local coordinates on M and the fibres of T^*M respectively. Now consider a Lagrangian submanifold $\iota : L \hookrightarrow T^*M$ with respect to the standard symplectic form $\omega := \mathrm{d}q_i \wedge \mathrm{d}x^i$.

Proposition A.0.2 (Locally-a-Section iff Local Diffeomorphism)
A Lagrangian submanifold $\iota : L \hookrightarrow T^*M$ is locally-a-section if and only if the map $\pi|_L := \pi \circ \iota : L \to M$ is a local diffeomorphism.

Proof. Let L be locally-a-section, so for all $y \in L$, there exists $V \subseteq L$ open, $U \subseteq M$ open, and $\psi \in \mathscr{C}^\infty(U)$ such that $\mathrm{d}\psi(U) = \iota(V)$. We wish to show that $\pi|_L(V)$ is open in M and $\pi|_L|_V := \pi|_V = \pi|_{\iota(V)} \circ \iota$ is a diffeomorphism onto its image. As (L, ι) is a smoothly embedded submanifold, ι is a topological diffeomorphism, hence V and $\iota(V)$ are diffeomorphic, with $\iota(V)$ carrying the subspace topology in T^*M. Further, as the restriction $\pi|_{\mathrm{d}\psi(U)}$ is the inverse of $\mathrm{d}\psi : U \to \mathrm{d}\psi(U) \subseteq T^*M$, U and $\mathrm{d}\psi(U)$ are diffeomorphic, where $\mathrm{d}\psi(U)$ carries the subspace topology in T^*M. Combining this with the locally-a-section Definition A.0.1 property, $\mathrm{d}\psi(U) = \iota(V)$, one has that the composition $\pi|_V$ is a diffeomorphism onto image, with inverse $\iota^{-1} \circ \mathrm{d}\psi :$

$U = \pi_L(V) \to V$. Note that $\pi|_L(V) = U$, which is known to be open in M. In summary, $\pi|_L$ is a local diffeomorphism.

Conversely, let y be an arbitrary point in L. By the local diffeomorphism property of $\pi|_L : L \to M$, there exists some open neighbourhood $V \subseteq L$ of y such that $\pi|_L(V) =: U$ is open in M and $\pi|_V$ is a diffeomorphism onto its image. Let y^i be local coordinates on V. Then, $V \ni y^i \overset{\pi|_L}{\longmapsto} x^i(y) \in U$. Again by the local diffeomorphism properties of $\pi|_L$, we have locally the invertibility of the Jacobian $\frac{\partial x^i}{\partial y^j}$ and the inverse relation $y^i = y^i(x)$. Hence, the embedding $\iota : L \hookrightarrow T^*M$ becomes $i : y^i \mapsto (x^i(y), q_i(y)) = (x^i, q_i(y(x))) =: (x^i, p_i(x))$ in local coordinates. Furthermore, since L is Lagrangian with respect to $\omega = \mathrm{d}q_i \wedge \mathrm{d}x^i$, we find

$$\frac{\partial x^i}{\partial y^j}\frac{\partial q_i}{\partial y^k} = \frac{\partial x^i}{\partial y^k}\frac{\partial q_i}{\partial y^j} \tag{A.0.1}$$

upon computing $\iota^*\omega = 0$.[1] Hence,

$$\begin{aligned}
\frac{\partial x^i}{\partial y^l}\left(\frac{\partial p_j}{\partial x^i} - \frac{\partial p_i}{\partial x^j}\right) &= \frac{\partial x^i}{\partial y^l}\left(\frac{\partial y^k}{\partial x^i}\frac{\partial q_j}{\partial y^k} - \frac{\partial y^k}{\partial x^j}\frac{\partial q_i}{\partial y^k}\right) \\
&= \frac{\partial q_j}{\partial y^l} - \frac{\partial y^k}{\partial x^j}\frac{\partial x^i}{\partial y^l}\frac{\partial q_i}{\partial y^k} \\
&= \frac{\partial q_j}{\partial y^l} - \frac{\partial y^k}{\partial x^j}\frac{\partial x^i}{\partial y^k}\frac{\partial q_i}{\partial y^l} \\
&= \frac{\partial q_j}{\partial y^l} - \frac{\partial q_j}{\partial y^l} \\
&= 0\,.
\end{aligned} \tag{A.0.2}$$

Therefore,

$$\frac{\partial p_j}{\partial x^i} - \frac{\partial p_i}{\partial x^j} = 0 \tag{A.0.3}$$

that is, the one-form $\eta := p_i \mathrm{d}x^i$ is closed. Consequently, by the Poincaré lemma, there is a function $\psi \in \mathscr{C}^\infty(U)$ so that $\eta = \mathrm{d}\psi$ (and therefore $p_i = \partial_i\psi$). It follows that $\mathrm{d}\psi(U) = \iota(V)$. In summary, L is locally-a-section. □

Note that the forward implication that $\pi|_L$ is a local diffeomorphism if the Lagrangian submanifold $\iota : L \hookrightarrow M$ is locally a section does not require any contractibility assumption on M, in contrast to the converse direction, which utilises the Poincarè lemma. Observe from the above proof that, for locally-a-section submanifolds and suitably nice sets $V \subseteq L$, $U = \pi|_L(V) \subseteq M$, one can explicitly write the inverse $\iota^{-1} \circ \mathrm{d}\psi : U \to V$ of the map $\pi|_L$. Consequently, by the local diffeomorphism property of $\pi|_L$, one may use the same coordinates on such suitably nice corresponding pairs of sets. Conversely, if a Lagrangian submanifold L has an open neighbourhood around each point on which one can take coordinates from M, it follows that the submanifold is locally-a-section.

[1] Note that $\iota_*\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j} + \frac{\partial q_j}{\partial y^i}\frac{\partial}{\partial q_j}$.

B

Geometric Structures

B.1 Almost Complex Structures and Differential Forms

Let (N, ω) be a $2m$-dimensional almost symplectic manifold (that is, ω is non-degenerate but not necessarily closed). Following [14], a differential p-form is called ω-effective if and only if $\omega^{-1} \lrcorner \alpha = 0$. Whenever $p = m$, this is equivalent to requiring $\alpha \wedge \omega = 0$, as defined in (2.1.1) for $N = T^*M$. Then we have the Hodge–Lepage–Lychagin theorem [14] (see also the text book [13] for a comprehensive treatment), stated as follows:

Theorem B.1.1 (Hodge–Lepage–Lychagin Theorem)
Let (N, ω) be an almost symplectic manifold. Then, any differential p-form $\alpha \in \Omega^p(N)$ has a unique decomposition $\alpha = \alpha_0 + \alpha_1 \wedge \omega + \alpha_2 \wedge \omega \wedge \omega + \cdots$ into ω-effective differential $(p-2k)$-forms $\alpha_k \in \Omega^{p-2k}(N)$. Furthermore, if two ω-effective p-forms vanish on the same p-dimensional isotropic submanifolds, they must be proportional.

Let now N be four-dimensional and (ω, α) a Monge–Ampère structure on N, that is, $\alpha \in \Omega^2(N)$ with $\alpha \wedge \omega = 0$ and suppose that $\mathsf{Pf}(\alpha) \in \mathscr{C}^\infty(N)$, defined by $\alpha \wedge \alpha = \mathsf{Pf}(\alpha)\omega \wedge \omega$, is non-zero. We then set [15]

$$\frac{\alpha}{\sqrt{|\mathsf{Pf}(\alpha)|}} =: J_\alpha \lrcorner \omega \ , \tag{B.1.1}$$

where J_α is an almost complex (respectively, para-complex) structure when $\mathsf{Pf}(\alpha) > 0$ (respectively, $\mathsf{Pf}(\alpha) < 0$). The differential forms ω and $J_\alpha \lrcorner \omega$ define the non-degenerate differential $(2,0)$- and $(0,2)$-forms with respect to J_α.[1] Then, we have the following result:

Proposition B.1.2 (Existence of Differential $(1,1)$-Forms)
For J_α as defined in (B.1.1) there exists a differential $(1,1)$-form K on N such that $K \wedge K \neq 0$, $K \wedge \omega = 0$, and $K \wedge (J_\alpha \lrcorner \omega) = 0$.

[1] In particular, for J_α almost complex, fix $\omega^{(\pm)} = \omega \pm iJ_\alpha \lrcorner \omega$ and note $J_\alpha \lrcorner \omega^\pm = \mp i\omega^\pm$. For J_α almost para-complex, omitting the factor of i in $\omega^\pm$ yields $J_\alpha \lrcorner \omega^\pm = \pm\omega^\pm$.

Proof. Note that ω and $J_\alpha \lrcorner \omega$ are linearly independent. Next, let $\rho \in \Omega^2(N)$ be such that $\{\omega, J_\alpha \lrcorner \omega, \rho\}$ is linearly independent. By Theorem B.1.1, we have a unique decomposition $\rho = \rho_0 + \lambda_0\omega$ with $\rho_0 \wedge \omega = 0$ and $\lambda_0 \in \mathscr{C}^\infty(N)$. Since $(J_\alpha \lrcorner \omega) \wedge (J_\alpha \lrcorner \omega) \neq 0$, we may again apply Theorem B.1.1 to obtain the unique decomposition $\rho_0 = \rho_1 + \lambda_1(J_\alpha \lrcorner \omega)$ with $\lambda_1 \in \mathscr{C}^\infty(N)$ such that $\rho_1 \wedge (J_\alpha \lrcorner \omega) = 0$. Since $(J_\alpha \lrcorner \omega) \wedge \omega = 0$, we also have $\rho_1 \wedge \omega = 0$. Hence, $\{\omega, J_\alpha \lrcorner \omega, \rho_1\}$ is linearly independent, and we must also have that $\rho_1 \wedge \rho_1 \neq 0$ since the exterior product yields a non-degenerate metric on $\bigwedge^2 T^*N$. In summary, we have thus obtained a $K := \rho_1$ such that $K \wedge K \neq 0$, $K \wedge \omega = 0$, and $K \wedge (J_\alpha \lrcorner \omega) = 0$. Finally, since ω and $J_\alpha \lrcorner \omega$ combine to give the differential $(2,0)$-form $\Omega^{(2,0)}$ and differential $(0,2)$-form $\Omega^{(0,2)}$ and since $K \wedge \omega = 0$ and $K \wedge (J_\alpha \lrcorner \omega) = 0$, we conclude that $K \wedge \Omega^{(2,0)} = 0$ and $K \wedge \Omega^{(0,2)} = 0$. Since $\Omega^{(2,0)} \wedge \Omega^{(0,2)} \neq 0$, K must be of type $(1,1)$ with respect to J_α. □

B.2 Integrability and Quaternionic Structures

As promised in Section 3.2, we now demonstrate how to refine the almost (para-)Hermitian form (3.2.6) and almost (para-)Hermitian metric (2.2.5) in such a way that they induce a triple of endomorphisms exhibiting (pseudo-)quaternionic behaviour. In particular, we show that for constant, non-zero $\hat{f}$, the differential $(1,1)$-form $\hat{\mathcal{K}}$ is a hyper-(para-)Kähler structure, hence $\hat{g}$ is a hyper-(para-)Kähler form.

Let M be a two-dimensional Riemannian manifold, with differential forms α and φ defined on T^*M as in (3.3.1). Further, let ω be the standard symplectic form on T^*M. Following [11,7], consider the normalised form $\alpha_{\hat{f}} = \frac{\alpha}{\sqrt{|\hat{f}|}}$ and take the triple of differential 2-forms $(\omega, \varpi, \alpha_{\hat{f}})$. Note that both ω and ϖ are non-degenerate and closed, and we have the following relations

$$\alpha_{\hat{f}} \wedge \alpha_{\hat{f}} = \operatorname{sgn}(\hat{f})\omega \wedge \omega\,, \qquad \alpha_{\hat{f}} \wedge \alpha_{\hat{f}} = \operatorname{sgn}(\hat{f})\varpi \wedge \varpi\,, \qquad \varpi \wedge \varpi = \omega \wedge \omega\,, \tag{B.2.1}$$

$$\alpha_{\hat{f}} \wedge \omega = 0\,, \qquad \alpha_{\hat{f}} \wedge \varpi = 0\,, \qquad \varpi \wedge \omega = 0\,, \tag{B.2.2}$$

Observe that $\alpha_{\hat{f}}$ is non-degenerate where it is well defined, that is, when $\hat{f} \neq 0$. Further, since α is closed, it follows that $\mathrm{d}\alpha_{\hat{f}} = \mathrm{d}\left(\frac{1}{\sqrt{|\hat{f}|}}\right)\alpha$, hence $\alpha_{\hat{f}}$ is closed if and only if $\hat{f}$ is constant on T^*M. It follows that the triple (ω, ϖ, α) are pairwise Monge–Ampère structures. Recall from (2.2.3) and (3.2.4), it is possible to define the endomorphisms

$$\alpha_{\hat{f}} =: \hat{J} \lrcorner \omega \quad \text{and} \quad \alpha_{\hat{f}} =: \hat{\mathcal{J}} \lrcorner \varpi\,, \tag{B.2.3}$$

which are almost complex for $\hat{f} > 0$, almost para-complex for $\hat{f} < 0$, and by the Lychagin–Rubtsov theorem [15,13] are integrable precisely when $\alpha_{\hat{f}}$ is closed, that is, if and only if $\hat{f}$ is constant. We may additionally define the following endomorphism

$$\varpi =: \hat{\mathcal{R}} \lrcorner \omega\,, \tag{B.2.4}$$

which is always a complex structure, by (B.2.1). Further, recalling from (2.2.4) and (3.2.6) that

$$\hat{K} \;:=\; -\operatorname{sgn}(\hat{f})\sqrt{|\hat{f}|}\varpi \quad\text{and}\quad \hat{\mathcal{K}} \;:=\; \operatorname{sgn}(\hat{f})\sqrt{|\hat{f}|}\omega\,, \tag{B.2.5}$$

we construct a triple of almost (para-)Hermitian forms on T^*M, namely $(\hat{K}, \hat{\mathcal{K}}, \alpha)$. In addition, the metric (2.2.5) can then be described in each of the following ways

$$\hat{g}(X,Y) \;=\; \hat{K}(X,\hat{J}Y) \;=\; \hat{\mathcal{K}}(X,\hat{\mathcal{J}}Y) \;=\; \alpha(X,\hat{\mathcal{R}}Y)\,, \tag{B.2.6}$$

hence $\hat{g}$ is almost (para-)Hermitian with respect to all three of our almost complex structures when $\hat{f} \neq 0$ and (para-)Kähler precisely when $\hat{f}$ is a non-zero constant. It can be verified that the compositions of $\hat{\mathcal{R}}$, $\hat{J}$, and $\hat{\mathcal{J}}$ obey the following Cayley table

	I	$\hat{\mathcal{R}}$	$\hat{J}$	$\hat{\mathcal{J}}$
I	I	$\hat{\mathcal{R}}$	$\hat{J}$	$\hat{\mathcal{J}}$
$\hat{\mathcal{R}}$	$\hat{\mathcal{R}}$	$-I$	$\hat{\mathcal{J}}$	$-\hat{J}$
$\hat{J}$	$\hat{J}$	$-\hat{\mathcal{J}}$	$-\operatorname{sgn}(\hat{f})I$	$\operatorname{sgn}(\hat{f})\hat{\mathcal{R}}$
$\hat{\mathcal{J}}$	$\hat{\mathcal{J}}$	$\hat{J}$	$-\operatorname{sgn}(\hat{f})\hat{\mathcal{R}}$	$-\operatorname{sgn}(\hat{f})I$

where I denotes the identity endomorphism. Hence, when $\hat{f} > 0$ (resp. $\hat{f} < 0$), the triple $(\hat{\mathcal{R}}, \hat{J}, \hat{\mathcal{J}})$ are (pseudo-)quaternionic [5], and $(\alpha, \varpi_{\hat{f}}, \hat{\mathcal{K}})$ form a triple of almost hyper (para-)complex structures. When $\hat{f}$ is constant in addition to the above, such that our structures are integrable, they are said to be hyper (para-)complex, and the metric $\hat{g}$ is not only (para-)Kähler, but hyper-(para-)Kähler.

In comparison to the hyper-(para-)Kähler structures that arise from the semi-geostrophic equations in [58] and the group structure of the analogous tensors presented in [11], the group structure of our almost (para-)complex structures exhibits a type change under the sign change of $\hat{f}$, which is not present in the aforementioned works. This suggests that our choice of Monge–Ampère structure is more natural for treating the dominance of vorticity and strain.

We make one final observation arising from the above structure. Observe that when $\hat{f}$ is a non-zero constant, T^*M is hyper-(para-)Kähler and hence (para-)Calabi-Yau. It then follows that T^*M with the metric $\hat{g}$ is Ricci flat. Ricci flatness implies that $\hat{R} = 0$ and using the formula (C.2.12b), specialised to the case of two-dimensional flows with constant $\hat{f}$, yields

$$0 \;=\; \hat{R} \;=\; \tfrac{\mathring{R}}{\hat{f}^2}\big(2\hat{f} - \mathring{\Delta}_{\mathrm{B}}p\big)\,. \tag{B.2.7}$$

It follows that either $\mathring{R} = 0$ or $\hat{f} = \frac{1}{2}\mathring{\Delta}_{\mathrm{B}}p$. The latter implies that $\mathring{R}|q|^2 = 0$ and as $|q|$ takes all non-negative real values on T^*M, from which it again follows that $\mathring{R} = 0$. Hence, for two-dimensional, incompressible flows, $\hat{f}$ being constant implies that M is flat. By contraposition then, M not being flat implies that $\hat{f}$ is non-constant, which in turn implies that $\hat{J}$ and $\hat{\mathcal{J}}$ are not integrable, that is, they are never (para-)complex on non-flat M.

C
Connections and curvatures

C.1 Pullback Metric in Two Dimensions

In what follows, we provide some more details on the computation of the Levi-Civita connection and Ricci curvature scalar associated with the metric (2.2.7) from Section 2.2. Firstly, recall that using (2.2.6), the metric (2.2.7) can be written in the form

$$g_{ij} \;=\; \zeta\tilde{g}_{ij} \quad \text{with} \quad \tilde{g}_{ij} \;=\; \psi_{ij} \;, \tag{C.1.1}$$

where the indices on $\psi \in \mathscr{C}^\infty(M)$ are interpreted via (2.2.10). Observe that, when $\zeta \neq 0$, g is a conformal scaling of the metric $\mathrm{sgn}(\zeta)\tilde{g}$ with conformal factor $|\zeta|$, where $\tilde{g}$ is the Hessian metric with respect to ψ. We wish to exploit this conformal nature to write the Levi-Civita connection and Ricci curvature scalar of g in terms of those of $\tilde{g}$.

C.1.1 Levi-Civita Connection of the Pullback Metric

We begin by observing that a consequence of the Ricci identity

$$[\mathring{\nabla}_i, \mathring{\nabla}_j]\eta_k \;=\; -\mathring{R}_{i(jk)}{}^l\eta_l \,, \tag{C.1.2}$$

for one-forms $\eta = \eta_i \mathrm{d}x^i$, is the following expression for the triple derivative of ψ in terms of the totally symmetrised triple derivative and curvature terms:

$$\mathring{\nabla}_i\psi_{jk} \;=\; \psi_{ijk} + \tfrac{1}{3}\big([\mathring{\nabla}_i, \mathring{\nabla}_j]\psi_k + [\mathring{\nabla}_i, \mathring{\nabla}_k]\psi_j\big) \;=\; \psi_{ijk} - \tfrac{2}{3}\mathring{R}_{i(jk)}{}^l\psi_l \,. \tag{C.1.3}$$

Upon applying the first Bianchi identity, we then find

$$\mathring{\nabla}_i\psi_{jk} + \mathring{\nabla}_j\psi_{ik} - \mathring{\nabla}_k\psi_{ij} \;=\; \psi_{ijk} + \tfrac{4}{3}\mathring{R}_{k(ij)}{}^l\psi_l \;. \tag{C.1.4}$$

Consequently, the Christoffel symbols for $\tilde{g}$ are given by

$$\begin{aligned}\tilde{\Gamma}_{ij}{}^{k} &= \tfrac{1}{2}\tilde{g}^{kl}(\partial_i\tilde{g}_{jl} + \partial_j\tilde{g}_{il} - \partial_l\tilde{g}_{ij}) \\ &= \mathring{\Gamma}_{ij}{}^{k} + \tfrac{1}{2}\tilde{g}^{kl}\big(\mathring{\nabla}_i\psi_{jl} + \mathring{\nabla}_j\psi_{il} - \mathring{\nabla}_l\psi_{ij}\big) \\ &= \mathring{\Gamma}_{ij}{}^{k} + \tfrac{1}{2}\Upsilon_{ijl}\tilde{g}^{lk}\ ,\end{aligned} \tag{C.1.5a}$$

where we have used (C.1.4) and introduced the notation

$$\Upsilon_{ijk} := \psi_{ijk} + \tfrac{4}{3}\mathring{R}_{k(ij)}{}^{l}\psi_l\ . \tag{C.1.5b}$$

This thus verifies (2.2.12b). Note that, in general, when the metric is changed by an overall sign, the Christoffel symbols of the second kind are unchanged, hence (C.1.5a) are also the Christoffel symbols for $\mathrm{sgn}(\zeta)\tilde{g}$, when $\zeta \neq 0$. The Christoffel symbols (2.2.12a) are then a result of the following proposition (see e.g. [59]) with $\phi = \frac{1}{2}\log(|\zeta|)$ and $g' = \mathrm{sgn}(\zeta)\tilde{g}$.

Proposition C.1.1 (Conformal Scaling of the Levi-Civita Connection)
Let M be a smooth (pseudo-)Riemannian manifold with metric $g = e^{2\phi}g'$, where $\phi \in \mathscr{C}^\infty(M)$ and g' is another metric on M, to which g is conformal. Let $\Gamma'_{ij}{}^{k}$ denote the Christoffel symbols of the second kind associated with g'. Then the Christoffel symbols of the second kind associated with g are given in terms of g' as

$$\Gamma_{ij}{}^{k} = \Gamma'_{ij}{}^{k} + (\partial_i\phi)\delta_j{}^{k} + (\partial_j\phi)\delta_i{}^{k} - (\partial_l\phi)g'^{lk}g'_{ij}\,, \tag{C.1.6}$$

where g'^{ij} denotes the inverse of the metric g'_{ij}.

C.1.2 Ricci Curvature Scalar of the Pullback Metric

Let us now turn to computing the curvature scalar for the metric (2.2.7). Firstly, we note that

$$\begin{aligned}\tilde{R}_{ijk}{}^{l} &= \partial_i\tilde{\Gamma}_{jk}{}^{l} - \partial_j\tilde{\Gamma}_{ik}{}^{l} - \tilde{\Gamma}_{ik}{}^{p}\tilde{\Gamma}_{jp}{}^{l} + \tilde{\Gamma}_{jk}{}^{p}\tilde{\Gamma}_{ip}{}^{l} \\ &= \mathring{R}_{ijk}{}^{l} + \tfrac{1}{2}\big(\mathring{\nabla}_i\Upsilon_{jk}{}^{l} - \mathring{\nabla}_j\Upsilon_{ik}{}^{l} - \tfrac{1}{2}\Upsilon_{ik}{}^{p}\Upsilon_{jp}{}^{l} + \tfrac{1}{2}\Upsilon_{jk}{}^{p}\Upsilon_{ip}{}^{l}\big)\,,\end{aligned} \tag{C.1.7}$$

where we have used (C.1.5a) and set $\Upsilon_{ij}{}^{k} := \Upsilon_{ijl}\tilde{g}^{lk}$. Next, using (C.1.3), it is straightforward to show that

$$\mathring{\nabla}_i\tilde{g}^{jk} = -\tilde{g}^{jl}\tilde{g}^{kp}\big(\psi_{ilp} - \tfrac{2}{3}\mathring{R}_{i(lp)}{}^{n}\psi_n\big)\,. \tag{C.1.8}$$

and

$$\mathring{\nabla}_i\psi_{jkl} = \psi_{ijkl} - \tfrac{3}{2}\mathring{R}_{i(jk}{}^{p}\psi_{l)p}\ . \tag{C.1.9}$$

Using these two relations, we find that

$$\begin{aligned}\mathring{\nabla}_i\Upsilon_{jk}{}^{l} &= \mathring{\nabla}_i(\tilde{g}^{lp}\Upsilon_{jkp}) \\ &= -\tilde{g}^{lr}\big(\psi_{irs} - \tfrac{2}{3}\mathring{R}_{i(rs)}{}^{n}\psi_n\big)\Upsilon_{jk}{}^{s} \\ &\quad + \tilde{g}^{lp}\big(\psi_{ijkp} - \tfrac{3}{2}\mathring{R}_{i(jk}{}^{n}\psi_{p)n} + \tfrac{4}{3}\mathring{R}_{p(jk)}{}^{n}\psi_{in} + \tfrac{4}{3}\mathring{\nabla}_i\mathring{R}_{p(jk)}{}^{n}\psi_n\big)\,.\end{aligned} \tag{C.1.10}$$

Upon substituting this expression and (C.1.5b) into (C.1.7), the curvature scalar (2.2.13b) then follows directly from the traces $\tilde{R} = \tilde{g}^{ij}\tilde{R}_{kij}{}^k$.

The following proposition (see e.g. [59]), again with $\phi = \frac{1}{2}\log(|\zeta|)$, gives the Ricci curvature scalar of g in terms of that for $g' = \mathrm{sgn}(\zeta)\tilde{g}$, when $m = 2$.

Proposition C.1.2 (Conformal Scaling of the Ricci Curvature Scalar)
Let M be a smooth, m-dimensional, (pseudo-)Riemannian manifold with metric $g = e^{2\phi}g'$, where $\phi \in \mathscr{C}^\infty(M)$ and g' is another metric on M, to which g is conformal. Let R' denote the Ricci curvature scalar associated with g'. Then the Ricci curvature scalar associated with g is given in terms of g' as

$$R \;=\; e^{-2\phi}\left[R' - 2(m-1)\Delta'\phi - (m-2)(m-1)|\mathrm{d}\phi|^2\right] , \tag{C.1.11}$$

where g'^{ij} denotes the inverse of the metric g'_{ij} and Δ' denotes the Beltrami Laplacian with respect to g'.

Recall that the Beltrami Laplacian with respect to a metric $\tilde{g}$ can be written as

$$\tilde{\Delta}\phi = \frac{1}{\sqrt{|\det(\tilde{g})|}}\partial_i\left[\sqrt{|\det(\tilde{g})|}\tilde{g}^{ij}\partial_j\phi\right] , \tag{C.1.12}$$

and note that, when the metric is changed by an overall sign, both the Ricci curvature scalar and the Beltrami Laplacian change by the same sign. The curvature scalar (2.2.13a) then follows from applying these observations to (C.1.11), hence writing the Ricci curvature scalar of g in terms of $\tilde{g}$.

C.2 Lychagin–Rubtsov Metric with Arbitrary Background Dimension

We now wish to compute the curvature related to the metric (2.2.5). Before we do so, however, let us summarise some general formulæ from the vielbein formalism as it is more efficient than working in a coordinate basis.

C.2.1 Vielbein formalism

Let (M, g) be an m-dimensional (semi-)Riemannian manifold coordinatised by x^i with $i, j, \ldots = 1, \ldots, m$. Then, $g = \frac{1}{2}g_{ij}\mathrm{d}x^i \odot \mathrm{d}x^j$. We denote the vielbeins by $E_a \in \mathfrak{X}(M)$ for $a, b, \ldots = 1, \ldots, m$ with $E_a = E_a{}^i\partial_i$ and $(E_a{}^i) \in \mathscr{C}^\infty(M, \mathsf{GL}(d))$. Dually, we have $e^a \in \Omega^1(M)$ with $E_a \lrcorner\, e^b = \delta_a{}^b$, $e^a = \mathrm{d}x^i e_i{}^a$ with $(e_i{}^a) \in \mathscr{C}^\infty(M, \mathsf{GL}(d))$, and $E_a{}^i e_i{}^b = \delta_a{}^b$ and $e_i{}^a E_a{}^j = \delta_i{}^j$. The metric can then be written as $g = \frac{1}{2}e^b \odot e^a\eta_{ab}$ with $\eta_{ab} = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1)$.

The structure functions $C_{ab}{}^c \in \mathscr{C}^\infty(M)$ are given by

$$[E_a, E_b] \;=\; C_{ab}{}^c E_c , \tag{C.2.1a}$$

or, dually,

$$\mathrm{d}e^a \;=\; \tfrac{1}{2}e^c \wedge e^b C_{bc}{}^a\;. \tag{C.2.1b}$$

The torsion and curvature two-forms,

$$T^a \;=\; \tfrac{1}{2}e^c \wedge e^b T_{bc}{}^a \quad\text{and}\quad R_a{}^b \;=\; \tfrac{1}{2}e^d \wedge e^c R_{cda}{}^b\;, \tag{C.2.2a}$$

are defined by the Cartan structure equations

$$\mathrm{d}e^a - e^b \wedge \omega_b{}^a \;=:\; -T^a \quad\text{and}\quad \mathrm{d}\omega_a{}^b - \omega_a{}^c \wedge \omega_c{}^b \;=:\; -R_a{}^b\;, \tag{C.2.2b}$$

where $\omega_a{}^b = e^c\omega_{ca}{}^b$ is the connection one-form. The associated Ricci tensor and the curvature scalar are then given by

$$R_{ab} \;:=\; R_{cab}{}^c \quad\text{and}\quad R \;:=\; \eta^{ba}R_{ab}\;. \tag{C.2.2c}$$

Furthermore, metric compatibility amounts to requiring

$$\omega_{ab} \;=\; -\omega_{ba} \quad\text{with}\quad \omega_{ab} \;:=\; \omega_a{}^c\eta_{cb}\;. \tag{C.2.3}$$

The Levi-Civita connection follows upon imposing the metric compatibility $\omega_{ab} = -\omega_{ba}$ as well as the torsion freeness $T^a = 0$, and a short calculation shows that it is given by

$$\omega_{ab}{}^c \;=\; \tfrac{1}{2}(C^c{}_{ab} + C^c{}_{ba} + C_{ab}{}^c) \tag{C.2.4}$$

with indices raised and lowered using η_{ab}. In this case, the curvature scalar (C.2.5) is

$$R \;=\; 2E_aC^a{}_b{}^b - C_{ab}{}^bC^a{}_c{}^c - \tfrac{1}{2}C_{abc}C^{acb} - \tfrac{1}{4}C_{abc}C^{abc}\;. \tag{C.2.5}$$

C.2.2 Levi-Civita Connection of the Lychagin–Rubtsov Metric

Let now $(M, \mathring{g})$ be an m-dimensional Riemannian manifold, and consider the the metric (3.3.6) on T^*M. Furthermore, let

$$\mathring{E}_a \;:=\; \mathring{E}_a{}^i\frac{\partial}{\partial x^i} \quad\text{and}\quad \mathring{e}^a \;:=\; \mathrm{d}x^i\mathring{e}_i{}^a \tag{C.2.6}$$

be the vielbeins and dual vielbeins on $(M, \mathring{g})$ with structure functions $\mathring{C}_{ab}{}^c$, and set

$$\begin{gathered}(\hat{e}^A) \;=\; (\hat{e}^a, \hat{e}_a) \;:=\; \left(\sqrt{|\hat{f}|}\,\mathrm{d}x^i\mathring{e}_i{}^a, \mathring{E}_a{}^i\,\mathring{\nabla}q_i\right),\\ (\hat{\eta}_{AB}) \;=\; \begin{pmatrix}\hat{\eta}_{ab} & \hat{\eta}_a{}^b\\ \hat{\eta}^a{}_b & \hat{\eta}^{ab}\end{pmatrix} \;:=\; \begin{pmatrix}\mathrm{sgn}(\hat{f})\mathbb{1}_m & 0\\ 0 & \mathbb{1}_m\end{pmatrix},\end{gathered} \tag{C.2.7}$$

for multi-indices $A, B, \ldots$. Then, the metric (3.3.6) becomes

$$\hat{g} \;=\; \tfrac{1}{2}\hat{e}^B \odot \hat{e}^A\hat{\eta}_{AB}\;. \tag{C.2.8}$$

Note that $\mathring{e}_i{}^a$ and $\mathring{E}_a{}^i$ do only depend on the base manifold coordinates x^i and not on the fibre coordinates q_i. Next, dually, we have $\hat{E}_A \lrcorner \hat{e}^B = \delta_A{}^B$ with $(\hat{E}_A) = (\hat{E}_a, \hat{E}^a)$ and

$$\hat{E}_a \;:=\; \frac{1}{\sqrt{|\hat{f}|}}\mathring{E}_a{}^i\left(\frac{\partial}{\partial x^i} + \mathring{\Gamma}_{ij}{}^k q_k \frac{\partial}{\partial q_j}\right) \quad \text{and} \quad \hat{E}^a \;:=\; \mathring{e}_i{}^a \frac{\partial}{\partial q_i}\;. \tag{C.2.9}$$

A straightforward calculation then yields, for $[\hat{E}_A, \hat{E}_B] = \hat{C}_{AB}{}^C \hat{E}_C$, the relations

$$[\hat{E}_a, \hat{E}_b] \;=\; \tfrac{1}{\sqrt{|\hat{f}|}}\mathring{C}_{ab}{}^c \hat{E}_c - \hat{E}_{[a} \log\big(|\hat{f}|\big)\hat{E}_{b]} + \tfrac{1}{|\hat{f}|}\mathring{R}_{abc}{}^d q_d \hat{E}^c\,, \tag{C.2.10a}$$

$$[\hat{E}_a, \hat{E}^b] \;=\; \tfrac{1}{2}\hat{E}^b \log\big(|\hat{f}|\big)\hat{E}_a - \tfrac{1}{\sqrt{|\hat{f}|}}\mathring{\omega}_{ac}{}^b \hat{E}^c\,, \tag{C.2.10b}$$

$$[\hat{E}^a, \hat{E}^b] \;=\; 0\,, \tag{C.2.10c}$$

where we have set $q_a := \mathring{E}_a{}^i q_i$ and used the identities

$$\mathring{\omega}_{ab}{}^c \;=\; \mathring{E}_a{}^i \mathring{E}_b{}^j \left(\mathring{\Gamma}_{ij}{}^k \mathring{e}_k{}^c - \frac{\partial}{\partial x^i}\mathring{e}_j{}^c\right) \quad \text{and} \quad \mathring{R}_{abc}{}^d \;=\; \mathring{E}_a{}^i \mathring{E}_b{}^j \mathring{E}_c{}^k \mathring{R}_{ijk}{}^l \mathring{e}_l{}^d\;. \tag{C.2.10d}$$

Reading off the structure functions $\hat{C}_{AB}{}^C$ from these relations and using the formula (C.2.4), the Levi-Civita connection $\hat{\omega}_{AB}{}^C$ for the metric (3.3.6) in terms of the Levi-Civita connection $\mathring{\omega}_{ab}{}^c$ for the background metric $\mathring{g}$ is given by

$$\hat{\omega}_{AB}{}^C \;=\; \tfrac{1}{2}(\hat{C}^C{}_{AB} + \hat{C}^C{}_{BA} + \hat{C}_{AB}{}^C)\;. \tag{C.2.11}$$

C.2.3 Ricci Curvature Scalar of the Lychagin–Rubtsov Metric

Upon combining (C.2.10) and (C.2.11) with (C.2.5), the curvature scalar of the metric (3.3.6) is given by

$$\begin{aligned}\hat{R} \;=\; &\tfrac{1}{\hat{f}}\mathring{R} - \tfrac{1}{4\hat{f}^2}\mathring{R}_{abc}{}^d \mathring{R}^{abce} q_d q_e - (m-1)\hat{\Delta}_{\rm B}\log\big(|\hat{f}|\big) - \delta_{ab}\hat{E}^a\hat{E}^b\log\big(|\hat{f}|\big)\\ &+ \tfrac{\mathrm{sgn}(\hat{f})}{4}(m-1)(m-2)\delta^{ab}\hat{E}_a\log\big(|\hat{f}|\big)\hat{E}_b\log\big(|\hat{f}|\big)\\ &+ \tfrac{1}{4}m(m-3)\delta_{ab}\hat{E}^a\log\big(|\hat{f}|\big)\hat{E}^b\log\big(|\hat{f}|\big)\;,\end{aligned} \tag{C.2.12a}$$

where $\hat{\Delta}_{\rm B}$ is the Beltrami Laplacian for $\hat{g}$. Here, $\mathring{R}_{abc}{}^d$ is the Riemann curvature tensor for the background metric $\mathring{g}$ and $\mathring{R}$ the associated curvature scalar. In our coordinate basis, this becomes

$$\begin{aligned}\hat{R} \;=\; &\frac{1}{\hat{f}}\mathring{R} - \frac{1}{4\hat{f}^2}\mathring{R}_{ijk}{}^l \mathring{R}^{ijkm} q_k q_m - (m-1)\hat{\Delta}_{\rm B}\log\big(|\hat{f}|\big) - \mathring{g}_{ij}\frac{\partial^2}{\partial q_i \partial q_j}\log\big(|\hat{f}|\big)\\ &+ \frac{1}{4\hat{f}}(m-1)(m-2)\mathring{g}^{ij}\left(\frac{\partial}{\partial x^i} + \mathring{\Gamma}_{ik}{}^l q_l \frac{\partial}{\partial q_k}\right)\log\big(|\hat{f}|\big)\left(\frac{\partial}{\partial x^j} + \mathring{\Gamma}_{jm}{}^n q_n \frac{\partial}{\partial q_m}\right)\log\big(|\hat{f}|\big)\\ &+ \frac{1}{4}m(m-3)\mathring{g}_{ij}\frac{\partial}{\partial q_i}\log\big(|\hat{f}|\big)\frac{\partial}{\partial q_j}\log\big(|\hat{f}|\big)\;,\end{aligned} \tag{C.2.12b}$$

where we have used (C.2.9). This verifies (2.2.9) and (3.3.7).

Finally, we note that in the case of the flat background metric $\mathring{g}_{ij} = \delta_{ij}$, we have $\hat{f} = f = \frac{1}{2}\Delta p$ with Δ the standard Laplacian on $\mathbb{R}^m$ and so, the formula (C.2.12b) simplifies to

$$\hat{R} \;=\; \frac{m-1}{4f^3}\big[(6-m)\partial_i f \partial^i f - 4f\Delta f\big] \;. \tag{C.2.13}$$

D
Submitted Training Hours

As part of the PGR Doctoral Research Programme at the University of Surrey, there is a requirement of completing 100 hours of training through either internal masters courses, postgraduate courses via the MAGIC consortium, summer schools, or writing review abstracts for seminars attended throughout the year. I have completed 80 of my hours through undertaking MAGIC courses (listed below) and am claiming the remaining 20 hours from extended abstracts for two seminars I attended during my first semester.

Module Code	**Module**	**Number of Hours**	**Passed**
MAGIC002	Differential Topology and Morse Theory	10	Yes
MAGIC008	Lie Groups and Lie Algebras	20	Yes
MAGIC053	Sheaf Cohomology	10	Yes
MAGIC064	Algebraic Topology	20	Yes
MAGIC081	String Theory	10	Yes
MAGIC105	Symplectic Geometry	10	Yes

The remainder of this appendix consists of the two extended abstracts being submitted for training hours, as well as the associated seminar participation form (signed by an attending academic) and assessment form.

D.1 Seminar Review: Hyperbolic Angles in Lorentzian Pre-Length Spaces

This extended abstract is intended to give a brief overview of the talk entitled '*Hyperbolic Angles in Lorentzian Length Spaces*' given by Tobias Beran (Universität Wien) online on the 29th October 2021 as part of the DIANA Seminar series. The presentation was based on the paper [60] by Tobias Beran and collaborator Clemens Sämann, building on previous work by Michael Kunzinger, Clemens Sämann, and Roland Steinbauer in [61], [62], to which I refer the reader for a more extensive discussion of comparison theorems and causal curves.

Definition D.1.1 (Lorentzian Pre-length Space)
A set (X, d), equipped with a pre-order $\leqslant$ and a transitive relation $\ll$, is called a causal space if for all $x, y \in X$, the condition $x \ll y \to x \leqslant y$ holds (that is, $\ll$ is contained within $\leqslant$). If, in addition, X is equipped with a metric d and there exists a function $\tau : X \times X \Rightarrow [0, \infty]$ satisfying the following:

(i) τ is lower semi-continuous with respect to d,

(ii) The reverse triangle inequality $\tau(x, z) \geqslant \tau(x, y) + \tau(y, z)$ for all $x, y, z \in X$ with $x \leqslant y \leqslant z$,

(iii) $\tau(x, y) > 0 \Leftrightarrow x \ll y$ for all $x, y \in X$,

then the quintuple $(X, d, \tau, \leqslant, \ll)$ is called a Lorentzian pre-length space and τ is referred to as the time-separation function.

Note that throughout this appendix, $\leqslant$ will denote the pre-order in the Lorentzian pre-length space, while $\geqslant$ will denote the natural ordering on $\mathbb{R}$. Further, we refer to $\leqslant$ as the causal relation and $\ll$ as the timelike relation on X where nomenclature is necessary. Let $\mathbb{L}^2(K)$ denote the simply connected two-dimensional Lorentzian model space of constant (sectional) curvature $K \in \mathbb{R}$.[1]

Definition D.1.2 (Timelike and Comparison Triangles)
Let X be a Lorentzian pre-length space. We define the following:

(i) A timelike (geodesic) triangle $\Delta(x_1, x_2, x_3)$ in X is a triple of timelike related points $x_1 \ll x_2 \ll x_3 \in X$ such that there exists a (future-directed) causal curve of τ-length $\tau(x_i, x_j)$, which is finite, for all $i < j$.

(ii) A comparison triangle $\bar{\Delta}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{L}^2(K)$ for $\Delta(x_1, x_2, x_3)$ is a timelike triangle whose sides satisfy $\tau(x_i, x_j) = \bar{\tau}(\bar{x}_i, \bar{x}_j)$ for all $i < j$, where $\bar{\tau}$ is the time-separation function on $\mathbb{L}^2(K)$.[2]

[1] In particular, for $K = 0, 1, -1$ we have the Minkowski, de Sitter, and anti-de Sitter spaces respectively.

[2] The time-separation function on a Lorentzian model space is simply the Lorentzian metric on the space.

We shall call τ locally finite-valued on X if each point $x \in X$ has a neighbourhood $U \subseteq X$ such that $\tau|_{U\times U}$ s finite valued. Further, a timelike triangle $\Delta(x_1, x_2, x_3)$ is said to satisfy timelike size bounds for $K < 0$ if $\frac{\pi}{\sqrt{-K}} \geqslant \tau(x_1, x_3) \geqslant \tau(x_1, x_2) + \tau(x_2, x_3)$, where $\frac{\pi}{\sqrt{-K}}$ is called the finite timelike diameter of $\mathbb{L}^2(K)$.

Definition D.1.3 (K-comparison Angles)
Let $(X, d, \leqslant, \ll, \tau)$ be a Lorentzian pre-length space and $\Delta(x_1, x_2, x_3)$ a timelike triangle in X with comparison triangle $\bar{\Delta}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{L}^2(K)$, for some $K \in \mathbb{R}$. Should $K < 0$, impose that $\Delta(x_1, x_2, x_3)$ satisfies size bounds for K. The K-comparison (hyperbolic) angle at x_1 is then given by

$$\tilde{\measuredangle}^K_{x_1}(x_2, x_3) \;:=\; \measuredangle^{\mathbb{L}^2(K)}_{\bar{x}_1}(\bar{x}_2, \bar{x}_3)\,, \tag{D.1.1}$$

where the angle on the right hand side is the usual hyperbolic angle on the model space, given by the hyperbolic-cosine rule.

It is noted that the above construction is also defined for triangles where x_2 and x_3 are only causally related, that is, $x_2 \leqslant x_3$. The angle at x_3 may also be defined by appealing to time reversal. To define the angle at x_2, the finite side length $\tau(x_1, x_2)$ must be considered, as opposed to $\tau(x_2, x_1)$. To account for this, the angles at x_1 and x_3 are typically decorated with a minus sign. The concept of an angle can also be readily extended to timelike curves:

Definition D.1.4 (Upper Angles Between Curves)
For $(X, d, \leqslant, \ll, \tau)$ a Lorentzian pre-length space, with τ locally finite-valued, consider a pair of future (past) directed timelike curves $\alpha, \beta : [0, \epsilon) \to X$ with $\alpha(0) - \beta(0) =: x$. Denote by A_0 the set of pairs $(s, t) \in (0, \epsilon)^2$ on which either $\alpha(s) \leqslant \beta(t)$ or $\beta(t) \leqslant \alpha(s)$. The upper angle between α and β is given by

$$\measuredangle_x(\alpha, \beta) \;:=\; \limsup_{(s,t)\in\mathcal{A}_0\searrow 0} \tilde{\measuredangle}^0_x(\alpha(s), \beta(t))\,. \tag{D.1.2}$$

Upper angles always exist; if $\lim\limits_{(s,t)\in\mathcal{A}_0\searrow 0} \tilde{\measuredangle}^0_x(\alpha(s), \beta(t))$ exists and is finite, then we call $\measuredangle_x(\alpha, \beta)$ the angle between α and β.

Having defined the concept of an angle on a Lorentzian pre-length space, given a pair of timelike curves, the seminar closed by discussing in brief several additional results and open questions. Key among these included a triangle inequality for upper angles, local timelike curvature bounds defined in terms of the monotonicity of angles, and the concept of hinges. Such constructions may be useful in the application of synthetic Lorentzian geometry to the field of general relativity - in particular, they may be used to globalize curvature bounds on Lorentzian pre-length spaces, which shall be discussed a little at the end of the next section. It should be noted that the fundamental ideas above mirrored by own work during an LMS funded undergraduate research bursary.

D.2 Seminar Review: Gluing Constructions in Lorentzian Pre-Length Spaces

This second extended abstract covers material from an additional talk in the DIANA seminar series, following on from the above work and entitled *'Gluing Constructions for Lorentzian Length Spaces'*. The presentation was given on the 21st January 2022 by Felix Rott (Universität Wien) and treats the formulation of an analogue to the Reshetnyak theorem for the amalgamation of CAT(k) spaces (geodesic metric spaces with sectional curvature bounded above by $k \in \mathbb{R}$), in the setting of Lorentzian pre-length spaces.

Definition D.2.1 (Amalgamation of Metric Spaces)
Given metric spaces (X_1, d_1), (X_2, d_2) *with respective closed, isometric subspaces* A_1, A_2*, let* $\sim$ *be the equivalence relation given by the isometry* $f : A_1 \to A_2$ *such that* $a \sim f(a)$ *for all* $a \in A_1$*. Then the quotient space* $\tilde{X} := (X_1 \amalg X_2)/\sim$ *with metric denoted* $\tilde{d}$*, is called the amalgamation of* X_1 *and* X_2*.*

Theorem D.2.2 (Reshetnyak Theorem)
Consider now (X_1, d_1), (X_2, d_2) *a pair of proper, CAT(k) metric spaces with respective closed, complete, convex, and isometric subspaces* A_1, A_2*. Then the amalgamation* $(\tilde{X}, \tilde{d})$*, as defined above, is also a* $CAT(k)$ *space.*

It is worth giving an explicit description of the metric $\tilde{d}$ before proceeding. For two metric spaces $(X_1, d_1), (X_2, d_2)$, the disjoint union metric on $X := X_1 \amalg X_2$ is given by

$$d(x, y) := \begin{cases} d_i(x, y), & \text{if } x, y \in X_i \\ \infty, & \text{else.} \end{cases} \tag{D.2.1}$$

For a metric space (X, d) and an equivalence relation $\sim$, the quotient semi-metric $\tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty]$ on the quotient space $\tilde{X} := X/\sim$ is given by

$$\tilde{d}([x], [y]) := \inf\left\{ \sum_{i=1}^{n} d(x_i, y_i) \,\middle|\, x \sim x_1, \;\; y \sim y_n, \;\; x_{i+1} \sim y_i, \;\; n \in \mathbb{N} \right\}. \tag{D.2.2}$$

It follows that the metric on an amalgamation $\tilde{X} := (X_1 \amalg X_2)/\sim$ is given by the quotient semi-metric with respect to the disjoint union metric on $X_1 \amalg X_2$.

In order to replicate the above theorem in the Lorentzian pre-length space framework, we need to define the amalgamation of such spaces. In doing so, a time-separation function, causal relation, and timelike relation on the usual amalgamation of Lorentzian Pre-length spaces (which are metric spaces) should be defined and conditions on the resulting quintuple being a Lorentzian Pre-length space should be found. We proceed by noting the disjoint union of Lorentzian Pre-length spaces $(X_1, d_1, \leqslant_1, \ll_1, \tau_1), (X_2, d_2, \leqslant_2, \ll_2, \tau_2)$, is a Lorentzian Pre-length space

$(X_1 \amalg X_2, d, \leqslant, \ll, \tau)$, where d is the disjoint union metric, $\leqslant := \leqslant_1 \amalg \leqslant_2$, $\ll := \ll_1 \amalg \ll_2$,[1] and the disjoint union time separation is (in contrast with the disjoint union metric) defined as

$$\tau(x, y) \ := \ \begin{cases} \tau_i(x, y) & \text{if} x, y \in X_i \\ 0, & \text{else.} \end{cases} \tag{D.2.3}$$

The associated 'quotient time-separation function' on $\tilde{X} = X/\sim$ is then given by

$$\tilde{\tau}([x],[y]) \ := \ \sup\left\{ \sum_{i=1}^{n} d(x_i, y_i) \,\middle|\, x \sim x_1, \ \ y \sim y_n, \ \ x_{i+1} \sim y_i, \ \ x_i \leqslant y_i, \ \ n \in \mathbb{N} \right\} \tag{D.2.4}$$

from which causal $\tilde{\leqslant}$ and timelike $\tilde{\ll}$ relations may be defined

$$[x]\tilde{\leqslant}[y] \Leftrightarrow \left\{ \sum_{i=1}^{n} d(x_i, y_i) \,\middle|\, x \sim x_1, \ \ y \sim y_n, \ \ x_{i+1} \sim y_i, \ \ x_i \leqslant y_i, \ \ n \in \mathbb{N} \right\} \neq \varnothing \,, \tag{D.2.5a}$$

[2]

$$[x]\tilde{\ll}[y] \Leftrightarrow \tilde{\tau}([x],[y]) > 0 \,, \tag{D.2.5b}$$

respectively. It follows that $(\tilde{X}, \tilde{\leqslant}, \tilde{\ll})$ defined in this way is a causal space, however it is not true in general that $\tilde{\tau}$ is lower semi-continuous, so $(\tilde{X}, \tilde{d}, \tilde{\leqslant}, \tilde{\ll}, \tilde{\tau})$ may not be a Lorentzian pre-length space. Conditions under which $\tilde{\tau}$ is lower semi-continuous were discussed briefly as the presentation came to a close. In these cases, the Lorentzian pre-length space $(\tilde{X}, \tilde{d}, \tilde{\leqslant}, \tilde{\ll}, \tilde{\tau})$ is called the Lorentzian amalgamation of X_1 and X_2. A full analogue to the Reshetnyak gluing theorem in the synthetic Lorentzian case is then given by the above construction alongside said conditions - these latter notions are explored more thoroughly in the paper [63] on which the presentation was based.

Since attending the above pair of talks, I have undertaken further work with the speakers, to utilise the concepts of gluing constructions and angle comparison in the globalization of curvature bounds in Lorentzian pre-length space. A publication is currently under way, treating the synthetic Lorentzian analogue to the Alexandrov patchwork globalization theorem [64] for spaces with timelike curvature bounded above. Additionally, a Bonnet-Myers' theorem result has been determined for spaces of global curvature bounded below. Further analysis needs to be undertaken to globalize curvature bounded below, which shall hopefully be worked on in the coming months.

[1] Here $x \leqslant y$ iff there exists $i \in \{1, 2\}$ such that $x, y \in X_i$ and $x \leqslant_i y$ and similarly for $\ll$.

[2] Note we use the convention $\sup \varnothing = 0$

D.3 PGR Student Seminar Participation Form

Student Name: Lewis Napper
Supervisor Name: Dr. Martin Wolf, Prof. Ian Roulstone

Number of Hours Requested Towards Broadening Training (20 hours):

In the table below give the date of the talk, its title, the speaker's name and tick if you intend to submit an extended abstract for this talk. Ask an attending academic to initial the form to validate your attendance. Talks can include: colloquia, research group seminars, pre-viva talks, reading groups etc. by both internal and external speakers.

Date	**Abbreviated Title**	**Speaker**	**Extended Abstract?**	**Academic initials**
29/10/21	Hyperbolic Angles in LPLS	Tobias Beran	Yes	JDEG
21/1/22	Gluing Constructions in LPLS	Felix Rott	Yes	JDEG

To claim 10/20 hours towards your broadening training requires 1/2 extended abstracts and this form to be submitted with your Confirmation Report.

D.4 Assessment

To be completed by the Confirmation Examiners following a discussion of the extended abstract/s submitted by the student with the Confirmation Report.

1. Did the student submit the requisite number of extended abstracts for the requested number of hours? (1 for 10 hours or 2 for 20 hours)

   YES **NO**

2. Was the student able to suitably discuss the content of the seminars for which the extended abstracts were submitted?

   YES **NO**

3. In your opinion should the student be awarded the requested number of hours toward their broadening training commitment?

   YES **NO**

4. If you have further comments then add them here.

Signatures and Date:

Examiner 1: Examiner 2:

Date:

PGR Director:

Date:

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