(Higher) Monge–Ampère Geometry of the Navier–Stokes Equations

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Previous Talks in this Series

- Volodya Rubtsov (14th April 2023)
  Symplectic and Contact Geometry of Monge–Ampère Equations: Introduction and Applications. [Kushner et al. 2007]
- Roberto D'Onofrio (28th April 2023)
  Singularities in Geophysical Fluid Dynamics Through Monge–Ampère Geometry. [D'Onofrio et al. 2023]
- Ian Roulstone (2nd June 2023)
  Applications of Symplectic Geometry in Fluid Dynamics. [Banos et al. 2016]

Outline of Talk

1. Recap of Monge–Ampère Geometry
2. Vorticity and the Poisson Equation for Pressure
3. Monge–Ampère Geometry of 2D Incompressible Flows
4. (Higher) Monge–Ampère Geometry of 3D Incompressible Flows
5. Summary and Outlook

1. Recap of Monge–Ampère Geometry

(Contact) Monge–Ampère Equations

- Are non-linear second-order PDEs which are quasi-linear w.r.t second order partial derivatives, up to determinants of the Hessian or its minors.
- In two dimensions, they take the form

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0\,.$$

  where $A, B, \dots E$ can depend on $x, y, \psi, \psi_x, \psi_y$ non-linearly.
- If $A, B, \dots E$ do not depend on ψ, we have a symplectic Monge–Ampère (MA) equation.

Examples of Monge–Ampère Equations

Key linear examples:

- Laplace: $\Delta\psi = 0$
- Wave: $\Box\psi = 0$

From (3D) semi-geostrophic theory:

- Ertel: $\det(\mathrm{Hess}(P)) = q_g$
- Chynoweth–Sewell: $q_g(T_{xx}T_{yy} - (T_{xy})^2) + T_{zz} = 0$

Here, q_g is potential vorticity, P is a (modified) geopotential, and T is its partial Legendre dual with respect to x and y.

[Chynoweth and Sewell 1989, D'Onofrio et al 2023]

Terminology and Notation

- Configuration Space / Background: $\mathbb{R}^m$ (or M with metric g) with coordinates (x^i), $i = 1, 2, \cdots m$.
- Phase Space / Cotangent Bundle: $T^*\mathbb{R}^m$ (or T^*M) with coordinates (x^i, q_i), $i = 1, 2, \cdots m$ (q's are fibre coordinates).
- Use Einstein summation convention.

Monge–Ampère Structures and Solutions

- A Monge–Ampère Structure is a triple $(T^*\mathbb{R}^m, \omega, \alpha)$ with
  - ☞ $\omega \in \Omega^2(T^*\mathbb{R}^m)$ symplectic, e.g. $\omega = \mathrm{d}q_i \wedge \mathrm{d}x^i$,
  - ☞ $\alpha \in \Omega^m(T^*\mathbb{R}^m)$ is ω-effective, i.e. $\alpha \wedge \omega = 0$,

  We call α the Monge–Ampère Form. [Banos 2002]

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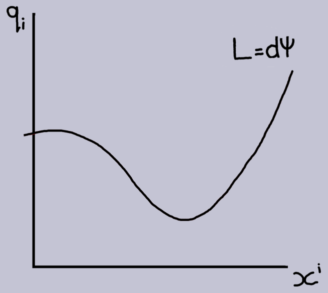
- A Generalised Solution to a MA equation, w.r.t. a MA structure, is a submanifold $L \hookrightarrow T^*\mathbb{R}^m$ s.t.
  - L is Lagrangian, i.e. $\dim(L) = m$ and $\omega|_L = 0$.
  - α vanishes on L, i.e. $\alpha|_L = 0$.

  [Kushner et al. 2007]

Recovering PDEs and Classical Solutions

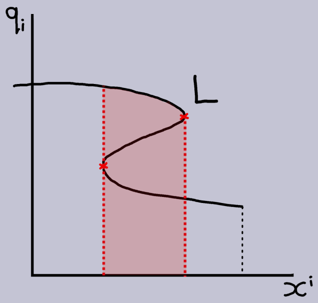
- Consider $L = \mathrm{d}\psi$ with coordinates $(x^i, \partial_i \psi)$ for some $\psi \in \mathscr{C}^\infty(\mathbb{R}^m)$.
- Trivially Lagrangian for canonical ω as $\omega|_{\mathrm{d}\psi} = 0$.
- The condition $\alpha|_{\mathrm{d}\psi} = 0$ corresponds to a MA equation, with classical solution ψ. [Lychagin 1979]
- The projection $\pi : L \to \mathbb{R}^m$ is a diffeomorphism.



More on Generalised Solutions

Pathologies of a generalised solution L:

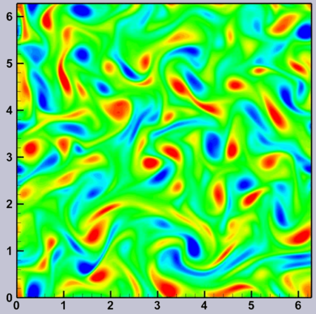
- When $\pi : L \to \mathbb{R}^m$ is not surjective (ψ is not defined on the whole domain).
- When $\pi : L \to \mathbb{R}^m$ is not injective (ψ is a multivalued solution). [Vinogradov 1973]
- When $\pi : L \to \mathbb{R}^m$ is not immersive — Arnold's Singularities.



2. Vorticity and the Poisson Equation for Pressure

The Importance of Vortices

- Turbulent flows consist of complex interactions of vortex structures.
- In 2D, they combine as they evolve, forming stable coherent structures characterised by circulation/elliptic motion.
- In 3D, one finds knotted/linked tubes which accumulate at small scale. “sinews of turbulence.” [Moffatt et al. 1994]



Vorticity of evolving 2d turbulence at early time (Andrey Ovsyannikov - Ecole Centrale de Lyon)

Pressure, Vorticity, and Strain

- Homogeneous, Incompressible Navier–Stokes on $\mathbb{R}^m$

$$\frac{\partial v^i}{\partial t} = -v^j \nabla_j v^i - \nabla^i p + \nu \Delta v^i \; (-c^i) \, .$$

$$\nabla_i v^i = 0$$

Pressure, Vorticity, and Strain

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$$\frac{\partial v^i}{\partial t} = -v^j \nabla_j v^i - \nabla^i p + \nu \Delta v^i \ (-c^i) \, .$$

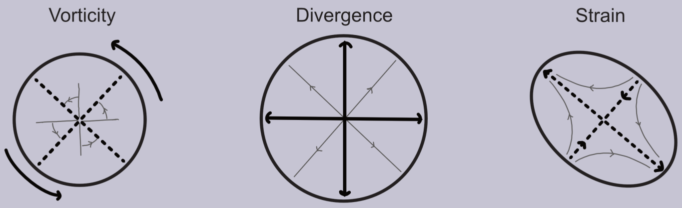
- Taking the divergence and applying $\nabla_i v^i = 0$ one finds

$$\Delta p \ (+\nabla_i c^i) = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} \, .$$

where $\zeta_{ij} = \frac{1}{2}(\nabla_j v_i - \nabla_i v_j)$ and $S_{ij} = \frac{1}{2}(\nabla_j v_i + \nabla_i v_j)$.

- Vorticity term dominates $\Leftrightarrow$ $\Delta p > 0$.
  Strain term dominates $\Leftrightarrow$ $\Delta p < 0$.

Vorticity, Divergence, and Strain



Based on Figure from Clough et al. 2014

$$(\zeta_{ij})_{2D} = \frac{1}{2}\begin{pmatrix} 0 & \zeta \\ -\zeta & 0 \end{pmatrix} \qquad (\zeta_{ij})_{3D} = \frac{1}{2}\begin{pmatrix} 0 & \zeta_3 & -\zeta_2 \\ -\zeta_3 & 0 & \zeta_1 \\ \zeta_2 & -\zeta_1 & 0 \end{pmatrix}$$

The Pressure Equation

(This) equation for the pressure is by no means fully understood and locally holds the key to the formation of vortex structures through the sign of the Laplacian of the pressure. In this relation... may lie a deeper knowledge of the geometry of both the Euler and Navier–Stokes equations.” [Gibbon 2008]

$$\Delta p = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij}$$

Pressure Equation in Two Dimensions

- In 2D, one has a stream function $v_1 = -\psi_y$ and $v_2 = \psi_x$.
- Pressure equation is a MA equation for the stream function

$$\frac{\Delta p}{2} = \left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) .$$

- *Vorticity dominates* $\Leftrightarrow$ $\Delta p > 0$ $\Leftrightarrow$ *Elliptic equation.*
  Strain dominates $\Leftrightarrow$ $\Delta p < 0$ $\Leftrightarrow$ *Hyperbolic equation.*
  No dominance $\Leftrightarrow$ $\Delta p = 0$ $\Leftrightarrow$ *Parabolic equation.*
  [Weiss 1991, Larchevêque 1993]

3. Monge–Ampère Geometry of $2D$ Incompressible Flows

Monge–Ampère Equations in Two Dimensions

The ω-effective MA forms for 2D background (4D phase space) are

$$\alpha = A\,\mathrm{d}q_1 \wedge \mathrm{d}x^2 + B\left(\mathrm{d}x^1 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}x^2\right) + C\,\mathrm{d}x^1 \wedge \mathrm{d}q_2 + D\,\mathrm{d}q_1 \wedge \mathrm{d}q_2 + E\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

.

Imposing that $\alpha|_{\mathrm{d}\psi} = 0$ yields ($x^1 = x$, $x^2 = y$, and $q_i = \partial_i \psi$)

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0$$

This correspondence is a bijection – unique MA form in ω-effective class.

Monge–Ampère Equations in Two Dimensions

- (Hodge–Lepage–Lychagin) Any $\beta \in \Omega^2(T^*\mathbb{R}^2)$ can be written

$$\beta = \alpha + F(x, q)\,\omega$$

  for symplectic form ω and some ω-effective form α.

- β is only ω-effective when $F \equiv 0$ so this is an equivalence relation.

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  for symplectic form ω and some ω-effective form α.

- β is only ω-effective when $F \equiv 0$ so this is an equivalence relation.
- If $\omega|_{\mathrm{d}\psi} = 0$, then $\alpha|_{\mathrm{d}\psi} = \beta|_{\mathrm{d}\psi}$ – They give the same equation!
- Effective forms remove redundancy from linear combinations (still have multiples of α giving equivalent equations).

Monge–Ampère Equations in Two Dimensions

- The Pfaffian is defined by $\alpha \wedge \alpha =: f\omega \wedge \omega$ where $f = AC - B^2 - DE$ is the determinant of the linearisation matrix for our PDE.

- Hence, the MA equation $\alpha|_{\mathrm{d}\psi} = 0$ is

  elliptic $\Leftrightarrow f > 0.$

  hyperbolic $\Leftrightarrow f < 0.$

  parabolic $\Leftrightarrow f = 0.$

The Lychagin–Rubtsov Theorem and Equivalence

- Define the endomorphism of vector fields $J : \mathfrak{X}(T^*\mathbb{R}^2) \to \mathfrak{X}(T^*\mathbb{R}^2)$ by

$$\frac{1}{\sqrt{|f|}}\alpha(\cdot\,,\cdot) =: \omega(J\,\cdot\,,\cdot)$$

- Almost complex $(J^2 = -1) \Leftrightarrow f > 0$
  Almost para-complex $(J^2 = +1) \Leftrightarrow f < 0$
  [Lychagin et al. 1993]

The Lychagin–Rubtsov Theorem and Equivalence

- Two MA forms (hence equations) α_1, α_2 are locally equivalent if there exists a local symplectomorphism $F : (T^*\mathbb{R}^2, \omega, \alpha_1) \to (T^*\mathbb{R}^2, \omega, \alpha_2)$ such that $F^*\alpha_2 = \alpha_1$.
- The Lychagin–Rubtsov theorem states t.f.a.e:
  - $\mathrm{d}(\frac{1}{\sqrt{|f|}}\alpha) = 0$.
  - $\alpha|_{\mathrm{d}\psi} = 0$ is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$.
  - J is integrable.

Geometry of the 2D Poisson Equation

One can recover the pressure equation

$$\frac{\Delta p}{2} = \left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right)$$

by choosing the MA form [Roulstone et al. 2009]

$$\alpha = \mathrm{d}q_1 \wedge \mathrm{d}q_2 - f\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\,,$$

with Pfaffian given by

$$f = \frac{\Delta p(x,y)}{2}\,.$$

The Pfaffian of the 2D Poisson Equation

- So $2f = \Delta p$ gives a geometric justification for the Poisson equation being:

$$\textit{elliptic} \Leftrightarrow \Delta p > 0.$$
$$\textit{hyperbolic} \Leftrightarrow \Delta p < 0.$$
$$\textit{parabolic} \Leftrightarrow \Delta p = 0.$$

- By Lychagin–Rubtsov Theorem,

$$\frac{\Delta p}{2} = (\psi_{xx}\psi_{yy} - \psi_{xy}^2)$$

is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$ iff Δp is constant.

The Lychagin–Rubtsov Metric

- For a choice of (non-degenerate, ω- and α-effective) $K \in \Omega^2(T^*\mathbb{R}^2)$, we can define a symmetric, bilinear form

$$\hat{g}(\cdot\,,\cdot) := -K(J\,\cdot\,,\cdot)$$

called the Lychagin–Rubtsov (LR) metric. [Roulstone et al. 2001]

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- There exists a choice of K s.t. the metric in (x^i, q_i) coordinates is

  $$\hat{g} = \begin{pmatrix} fI & 0 \\ 0 & I \end{pmatrix}$$

  with signature dictated by the sign of f.

LR Metric for the 2D Poisson Equation

The LR metric on $T^*\mathbb{R}^2$ given by

$$\hat{g} = \begin{pmatrix} \frac{\Delta p}{2} I & 0 \\ 0 & I \end{pmatrix}$$

is

$$\text{Riemannian} \Leftrightarrow \Delta p > 0.$$
$$\text{Kleinian} \Leftrightarrow \Delta p < 0.$$
$$\text{Degenerate} \Leftrightarrow \Delta p = 0.$$

N.B. These degeneracies correspond to singularities of the scalar curvature — they persist under coordinate changes.

Pull-back LR Metric for the 2D Poisson Equation

- The pull-back of the LR metric $\hat{g}$ to a classical solution $L = \mathrm{d}\psi$ is

$$\hat{g}|_{\mathrm{d}\psi} = \zeta \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix}$$

  where $\zeta = \Delta\psi$.

- Degenerate when $\zeta = 0$ or $\Delta p = 0$.
  Riemannian when $\Delta p > 0$.
  Kleinian when $\Delta p < 0$.

- Degeneracy when $\zeta = 0$ not always curvature singularity.

Geometry of the 2D Poisson Equation

Δp	> 0	< 0	$= 0$
Dominance	Vorticity	Strain	None
$\alpha\|_{\mathrm{d}\psi} = 0$	Elliptic	Hyperbolic	Parabolic
f	> 0	< 0	$= 0$
J^2	-1	1	Singular
$\hat{g}$	Riemannian $(4,0)$	Kleinian $(2,2)$	Degenerate
$\hat{g}\|_{\mathrm{d}\psi}$	Riemannian $(2,0)$	Kleinian $(1,1)$*	Degenerate

*Except when $\zeta = 0$, in which case it is degenerate.

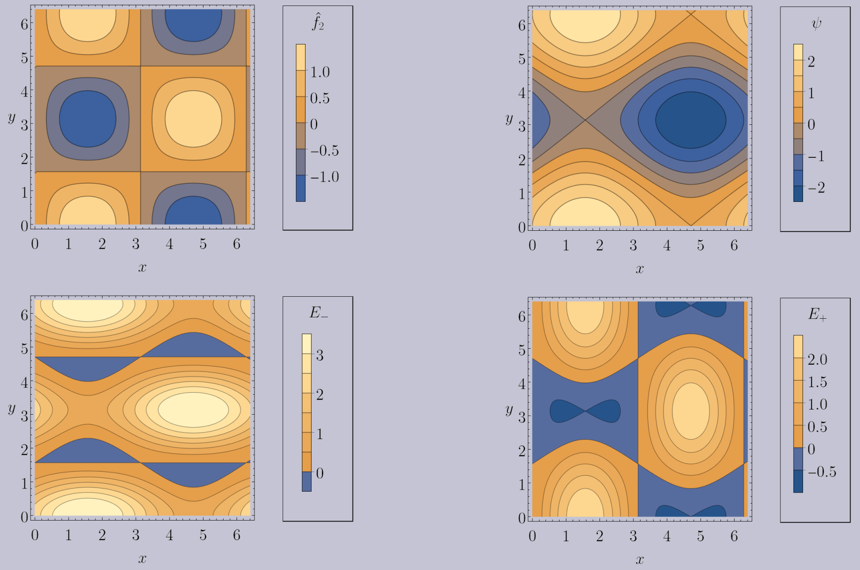
Topology of 2D Vortices

- For simply connected regions Σ of 2D flows on which $\Delta p > 0$ and with boundary given by a closed stream-line, all streamlines within Σ are also closed (and convex). [Larchevêque 1993]
- Σ is topologically a disc $[\chi(\Sigma) = \chi(\mathrm{d}\psi(\Sigma)) = 1]$ and Gauß–Bonnet theorem in $\mathrm{d}\psi(M)$ is:

$$\int_{\mathrm{d}\psi(\partial\Sigma)} \mathrm{d}s \; \kappa(x(s)) = 2\pi - \int_{\mathrm{d}\psi(\Sigma)} \mathrm{vol}_{\mathrm{d}\psi(\Sigma)} R$$

- The mean curvature of the boundary of a 'vortex' is described by gradients of vorticity and strain.

2D ABC Flow: $\psi(x,y) = \frac{3}{2}\cos(y) + \sin(x)$



4. (Higher) Monge–Ampère Geometry of 3D Incompressible Flows

Velocity and Divergence in 2D

- Rather than working with the stream function in $2\mathrm{D}$, work with velocity directly. Consider L with coordinates $(x^i, v_i(x))$.
- $\alpha|_L = 0$ gives Poission equation for pressure in terms of vorticity and strain, but now $\omega|_L = 0$ implies vanishing vorticity.
- Use a different symplectic form:

$$\begin{aligned}\varpi &= \mathrm{d}q_i \wedge \star\mathrm{d}x^i \\ &= \mathrm{d}q_1 \wedge \mathrm{d}x^2 - \mathrm{d}q_2 \wedge x^1\end{aligned}$$

whose pull-back to L gives $\nabla_i v^i = 0$.

Jacobi Systems in 2D

- Having $\alpha|_L = 0$ and $\varpi|_L = 0$ simultaneously is equivalent to

$$\begin{aligned} \nabla \cdot v &= 0 \\ \det(J(v, x)) &= \tfrac{1}{2}\Delta p \end{aligned}$$

- This is a Jacobi System – first order system of PDEs with nonlinearity given by determinant of Jacobian or its minors.
- These are studied in 2D where they generalise Monge–Ampère Equations and Cauchy–Riemann Systems.

Higher Monge–Ampère Problems/Jacobi Systems

- A k-Plectic Form is a closed and non-degenerate $\varpi \in \Omega^{k+1}(T^*\mathbb{R}^m)$. [Cantrijn et al. 2009]
- Consider structures of form $(T^*\mathbb{R}^m, \varpi, \alpha)$ where ϖ is $(m-1)$-plectic (no effectiveness condition).
- Generalised solutions are now submanifolds $L \hookrightarrow T^*\mathbb{R}^m$ satisfying $\varpi|_L = 0$ and $\alpha|_L = 0$ (not necessarily Lagrangian).
- We focus on L with coordinates $(x^i, v_i(x))$, diffeomorphic to $\mathbb{R}^m$, in lieu of classical solutions.

Almost (Para-)Complex Structure in 3D

- Original definition of J in 2D uses a pair of 2-forms. Need an alternative for higher dimensions, i.e. 3D.
- The Hitchin Isomorphism $\Phi : \Omega^5(T^*\mathbb{R}^3) \to \mathfrak{X}(T^*\mathbb{R}^3) \otimes \Omega^6(T^*\mathbb{R}^3)$ lets us define endomorphisms [Hitchin. 2000]

$$A : \mathfrak{X}(T^*\mathbb{R}^3) \to \mathfrak{X}(T^*\mathbb{R}^3); \quad A(X)\mathrm{vol} = \Phi(\iota_X \alpha \wedge \alpha)$$

for $X \in \mathfrak{X}(T^*\mathbb{R}^3)$ and choices of $\alpha \in \Omega^3(T^*\mathbb{R}^3)$ and vol.

- The Hitchin Pfaffian of α is then $f = \frac{1}{6}\,\mathrm{tr}(A^2)$ and $J := \frac{1}{\sqrt{|f|}} A$.

Lychagin–Rubtsov Metric in Higher Dimensions

- Again, for some choice of $K \in \Omega^2(T^*\mathbb{R}^m)$, we can define the Lychagin–Rubtsov metric

$$\hat{g}(\cdot\,,\cdot) := -K(J\cdot\,,\cdot)$$

- The analogous choice to 2D again gives

$$\hat{g} = \begin{pmatrix} f I_m & 0 \\ 0 & I_m \end{pmatrix}$$

in (x^i, q_i) coordinates, with signature dictated by the sign of the Hitchin Pfaffian f.

Metrics from Previous Seminars

- Results of [Banos. 2002] show that in 3D, for our choice of K, this is (conformally) equivalent to:

$$\hat{g}(X,Y) = \frac{(\iota_X \alpha) \wedge (\iota_X \alpha) \wedge \omega}{\omega^3}$$

- In the 2D literature, the metric

$$\hat{g} = \frac{2\left[(\iota_X \omega) \wedge (\iota_Y \alpha) + (\iota_Y \omega) \wedge (\iota_X \alpha)\right] \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2}{\omega \wedge \omega}$$

also appears, but is not in general equivalent to our choice of K.

Setup for Higher Dimensional Fluids

In higher dimensions, the forms ϖ, α from 2D generalise to

$$\varpi = \mathrm{d}q_i \wedge \star\, \mathrm{d}x^i$$
$$\alpha = \tfrac{1}{2}\mathrm{d}q_i \wedge \mathrm{d}q_j \wedge \star\left(\mathrm{d}x^i \wedge \mathrm{d}x^j\right) - \tfrac{1}{2}\Delta p\, \mathrm{vol}_m$$

When pulled back to L with coordinates $(x^i, v_i(x))$, they give

$$\nabla_i v^i = 0$$
$$\Delta p = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij}$$

the divergence free equation and Poisson equation respectively.

LR Metric for Higher Dimension Fluids

- The (Hitchin) Pfaffian is again $f = \frac{1}{2}\Delta p$ and the LR metric is

$$\hat{g} = \begin{pmatrix} \frac{\Delta p}{2} I_m & 0 \\ 0 & I_m \end{pmatrix} .$$

- For $A_{ij} = \nabla_j v_i$, the pullback metric is

$$(\hat{g}|_L)_{ij} = A^k{}_i A_{kj} - \tfrac{1}{2}\delta_{ij} A_{kl} A^{lk} .$$

- In general, signature change of $\hat{g}|_L$ does not coincide with sign change in f — more complicated relationship.

Topology of 3D Vortices

- No Gauss–Bonnet Theorem in odd dimensions – how to extract topological information?
- Let $\theta = q_i \mathrm{d}x^i$ be the tautological form. Then the helicity density is

$$(\theta \wedge \omega)|_L = v_i \zeta^i \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

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- Under ideal conditions, helicity is an invariant quantity and vorticity is conserved.
- Helicity can be related to topological quantities from knot theory i.e. the Gauss linking number, Călugăreanu invariant, and Jones Polynomial [Liu and Ricca 2012, Ricca and Moffatt 1992].

Extension to Riemannian Manifold

- On a Riemannian manifold (M, g), the approach is broadly the same:

$$\Delta p + R_{ij} v^i v^j \; (+\nabla_i c^i) = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} \, .$$

- Schematically take

$$\mathrm{d}q_i \to \mathrm{d}q_i - \mathrm{d}x^j \Gamma_{ij}{}^k q_k.$$
$$I \to g.$$
$$f = \tfrac{1}{2}\Delta p \to f = \tfrac{1}{2}(\Delta p + R^{ij} q_i q_j).$$

- Geometric justification for Weiss criterion for equation type still applies on a manifold, e.g. $\mathbb{S}^2$ [Napper et al. 2023].

Navier–Stokes equations in spherical geometry describe ocean/atmosphere dynamics (Joshua Stevens - NASA Earth Observatory)

3D Flows with Symmetry

- 2.5D Euclidean flows flows have velocity [Ohkitani et al. 2000]

$$v := \big(v_1(x^1,x^2,t), v_2(x^1,x^2,t), z\gamma(x^1,x^2,t) + W(x^1,y^2,t)\big)$$

- We have a 1D symmetry generated by
  - $\partial_{x^3} \in \mathfrak{X}(\mathbb{R}^3)$ when $\gamma \equiv 0$.
  - $\partial_{x^3} + \gamma\partial_{q_3} \in \mathfrak{X}(T^*\mathbb{R}^3)$ when $W = c\gamma$ for some $c \in \mathbb{R}$.

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- We have a 1D symmetry generated by
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  – $\partial_{x^3} + \gamma\partial_{q_3} \in \mathfrak{X}(T^*\mathbb{R}^3)$ when $W = c\gamma$ for some $c \in \mathbb{R}$.
- Shown that for Burgers' Vortex ($W \equiv 0$, $\gamma = \gamma(t)$), symplectic reduction reproduces Lundgren's Transformation and yields a 2D (compressible) flow [Banos et al. 2016].
- Explicitly extended to $\gamma \equiv 0$ in [Napper et al. 2023]

3D Flows with Symmetry

- These symmetries also apply to background manifolds with metric $g = g_2 + \mathrm{e}^{-2\varphi}\mathrm{d}x^3 \otimes \mathrm{d}x^3$ where $\varphi \in C^\infty(\mathbb{R}^3)$
- Symplectic reduction yields pressure and compressibility equations for v_1, v_2 in terms of ϖ and v_3, i.e.

$$\nabla_i v^i = -v^i \partial_i \varphi$$

and an LR metric on the reduced phase space $(T^*\mathbb{R}^2)$.

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  and an LR metric on the reduced phase space $(T^*\mathbb{R}^2)$.
- Also have access to 2-plectic reduction [Blacker. 2021] which directly gives v_i in terms of a stream function ψ

$$v_i = -\mathrm{e}^{-\varphi}\epsilon_{ij}\nabla^j \psi$$

5. Summary and Outlook

Summary

- We have recapped the concepts of Monge–Ampère structures and their associated geometry as a tool for studying MA PDEs.
- We applied this tool to the pressure equation in 2D and showed that the signatures of the LR metric and its pull-back act as diagnostics for equation type and the dominance of vorticity and strain.
- We discussed generalisations to higher dimensions and manifolds with curvature, providing geometric validation for Weiss-Okubo like criterion in these cases.

Outlook

- We looked at (locally) classical solutions – what happens if we allow fully generalised solutions with non-immersive projections?
- In semi-geostrophic theory, these produce degeneracy of $\hat{g}|_L$ and type change, which represent weather fronts. [D'Onofrio et al. 2023]
- The geometry of classical solutions models flows with elliptic vortices, vortex tubes, and lines. Perhaps singular locus of projections could be used to model vortex sheets.

Outlook

- Recall that we made a choice of differential form $K \in \Omega^2(T^*M)$ when defining the metric $\hat{g}$ (whole family of LR metrics)
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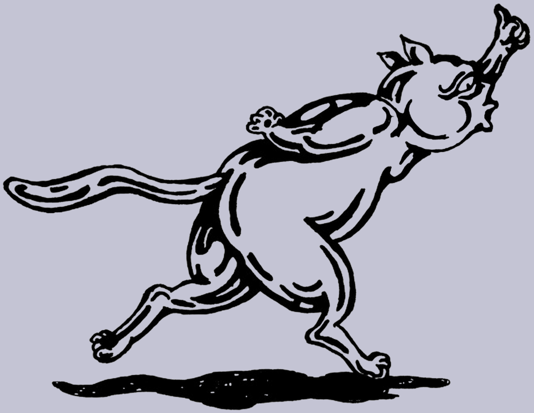
- In 2D, (ω, α, K) induce an almost hyper-complex triple [Roulstone, Rubtsov. 2001].
- For our choice of K, (anti-)self duality of the curvatures of $\hat{g}$ implies integrability properties of the triple and lets us map to twistor space [Ongoing work].
- The diagonal form of our LR metric reminds of the scaled Sasaki metrics, whose associated structures have been studied in detail. [Gezer et al. 2014].

Thank you!



Any questions?

(Image Credit [Kushner, Lychagin, Rubtsov. 2007])